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FUNCTIONAL ANALYSIS AND SEMI-GROUPS

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To
KIRSTI

And each man hears as the twilight nears,
to the beat of his dying heart,
The Devil drum on the darkened pane:
"You did it, but was it Art?"

FOREWORD

The analytical theory of semi-groups is a recent addition to the ever-growing list of mathematical disciplines. It was my good fortune to take an early interest in this discipline and to see it reach maturity. It has been a pleasant association: I hail a semi-group when I see one and I seem to see them everywhere! Friends have observed, however, that there are mathematical objects which are not semi-groups.

The present book is an elaboration of my Colloquium Lectures delivered before the American Mathematical Society at its August, 1944 meeting at Wellesley College. I wish to thank the Society and its officers for their invitation to present and publish these lectures.

The book is divided into three parts plus an appendix. My desire to give a practically self-contained presentation of the theory required the inclusion of an elaborate introduction to modern functional analysis with special emphasis on function theory in Banach spaces and algebras. This occupies Part One of the book and the Appendix; these portions can be read separately from the rest and may be used as a text in a course on operator theory. It is possible to cover most of the material in these six chapters in two terms.

The analytical theory of one-parameter semi-groups occupies Part Two while Part Three deals with the applications to analysis. The latter include such varied topics as trigonometric series and integrals, summability, fractional integration, stochastic theory, and the problem of Cauchy for partial differential equations. In the general theory the reader will also find an alternate approach to ergodic theory. All semi-groups studied in this treatise are referred to a normed topology; semi-groups without topology figure in a few places but no details are given. The task of developing an adequate theory of transformation semi-groups operating in partially ordered spaces is left to more competent hands.

The literature has been covered rather incompletely owing to recent war conditions and to the wide range of topics touched upon, which have made it exceedingly difficult to give the proper credits.

This investigation has been supported by grants from the American Philosophical Society and from Yale University which are gratefully acknowledged. On the personal side, it is a great pleasure to express my gratitude to the many friends who have aided me in preparing this book. J. D. Tamarkin, who read and criticized my early work in the field and who vigorously urged its inclusion in the Colloquium Series is beyond the reach of my gratitude. I am deeply indebted to Nelson Dunford and to Max Zorn who have contributed extensively to the book, the former chiefly to Chapters II, III, V, VIII, IX, and XIV, the latter to Chapters IV, VII, and XXII. Both have given me generously of their time and special experience. Various portions of the manuscript have been critically examined and amended by Warren Ambrose, E. G. Begle, H. Cramér, J. L. Doob, W. Feller, N. Jacobson, D. S. Miller, H. Pollard, C. E. Rickart, and I. E. Segal. To all helpers, named and unnamed, I extend my warmest thanks.

EINAR HILLE

New Haven, Conn.,
December, 1946

CONVENTIONS

Each Part of the book starts with a Summary, each Chapter with an Orientation. The chapters are divided into sections and the sections, except orientations, are grouped into paragraphs. Cross references are normally to sections, rarely to paragraphs. Section 3.10 is the tenth section of Chapter III; it belongs to §2 which is referred to as §3.2 when necessary. The page headings show the numbers of the current section and paragraph, the integral part of the former is the number of the chapter. Definitions, formulas, lemmas, and theorems are numbered within each section; thus Theorem 9.4.2 is the second theorem in section 9.4. References to the literature give the author's name followed by numerals in brackets referring to his book or paper by that number in the Bibliography at the close of the book. Such references are given in the text when needed; collected references for a whole chapter occur after the orientation to the chapter in question except in the case of chapters with heterogeneous subject matter when they are given at the end of each paragraph. There is a list of special symbols following the Index at the end of the book.

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PART ONE

FUNCTIONAL ANALYSIS

Summary. The first part of these Colloquium Lectures is devoted to a brief exposition of some of the basic ideas of modern functional analysis. There are five chapters entitled: *Abstract Spaces, Linear Operations, Vector Valued Functions, Functions on Vectors to Vectors,* and *Analysis in Banach Algebras.*

We start with a description of the topological and algebraical properties of the spaces to be considered later and proceed to a discussion of linear operations on one such space to another. The function theory proper starts in Chapter III where we are concerned with functions of real or complex variables, having their range in a Banach space; in Chapter IV the discussion turns to (analytic) functions having domain and range in complex Banach spaces. Finally in Chapter V either the domain or the range or both belong to a Banach algebra. For the algebraic side of the theory of Banach algebras as well as for further complements of the analytical theory, the reader is referred to the Appendix, Chapter XXII, *Notes on Banach Algebras.*

CHAPTER I

ABSTRACT SPACES

1.1. Orientation. This chapter is intended to give a brief review of the basic notions in the theory of abstract spaces which will be needed in the following. The material is grouped under four paragraph headings: *Topological Concepts*, *Additive Spaces*, *Linear Spaces*, and *Algebraic Spaces*. The abstract spaces occurring in this treatise normally have a definite algebraic structure in addition to being topological spaces of one type or another. This fact underlies the choice of the headings. As practically all the material is taken from current literature, proofs are usually omitted. The reader who needs further explanations is referred to the literature quoted at the end of the paragraphs.

1. Topological Concepts

1.2. Closure. Let $\mathfrak{X}$ be an *abstract space*, that is, a set whose elements x, y, $\cdots$ are called *points*. X, Y, X_α are subsets of $\mathfrak{X}$; the symbols $X \cup Y$ and $\bigcup_\alpha X_\alpha$ denote *unions*, $X \cap Y$ and $\bigcap_\alpha X_\alpha$ *intersections* of the sets occurring in the symbols. X and Y are *disjoint* if $X \cap Y = \emptyset$, the *empty set*. The *complement* of X with respect to $\mathfrak{X}$ is denoted by $\tilde{X}$.

A *closure topology* is introduced in $\mathfrak{X}$ by ordering to every set X in $\mathfrak{X}$ a set $\bar{X}$, called the *closure* of X, subject to the conditions:

C_1. $\overline{X \cup Y} = \bar{X} \cup \bar{Y}$,

C_2. $\bar{\bar{X}} = \bar{X}$,

C_3. $\bar{X} = X$ *if* X *is empty or is a single point.*

$\mathfrak{X}$ is then said to be a *topological space* in the sense of Kuratowski or a *T_1-space* in the sense of Alexandroff-Hopf. The axioms go back to M. Fréchet and F. Riesz.

The postulates imply that $X \subset \bar{X}$. If $X = \bar{X}$, the set X is *closed*, it is *open* if $\tilde{X}$ is closed. Closed sets have a closed intersection, open sets an open union. The union of a countable number of closed sets is called a set F_σ, the intersection of a countable number of open sets a set G_δ. An open set containing the point x is called a *neighborhood* of x and is denoted by $N(x)$ or N_x. The union of all open sets in X is the *interior* of X, denoted by Int (X). If $x \in$ Int (X), then x is an *interior point* of X. The *boundary* of X is the intersection of the closures of X and of $\tilde{X}$.

The point x_0 is a *limit point* of the set X if every $N(x_0)$ contains infinitely many points of X. The set X' of limit points of X is called the *derived set* of X and $\bar{X} = X \cup X'$. X is *closed* if $X' \subset X$, *dense in itself* if $X' \supset X$, *perfect* if $X' = X$.

A set X is *dense* in $\mathfrak{X}$ if $\bar{X} = \mathfrak{X}$, dense in Y if $Y \subset \overline{X \cap Y}$. X is a *separable set* if there is a countable set which is dense in X. In particular, $\mathfrak{X}$ is a *separable space* if there is a countable set which is dense in $\mathfrak{X}$. X is *nowhere dense* if $\text{Int}\,(\bar{X}) = \emptyset$.

X is of the *first category* in $\mathfrak{X}$ if X is the union of a countable number of sets each of which is nowhere dense in $\mathfrak{X}$, otherwise X is of the *second category*. It is of the *first category at the point* x if there exists a neighborhood $N(x)$ such that $N(x) \cap X$ is of the first category. It is of the *second category at* x if $N(x) \cap X$ is of the second category for every neighborhood $N(x)$ of x. The set of points of $\mathfrak{X}$ where a given set X is of the second category will be denoted by $D(X)$. We have $D(X) = \emptyset$ if and only if X is of the first category. The operation D is *additive* and *idempotent*, that is,

$$D(X \cup Y) = D(X) \cup D(Y), \quad D[D(X)] = D(X).$$

Finally, *$D(X)$ is closed, contained in the closure of X, and equals the closure of its own interior.*

X has the *property of Baire* if every non-void open set G contains a point where either X or $\tilde{X}$ is of the first category; such a set will become an open (or a closed) set upon adjunction and suppression of suitable sets of the first category.

A topological space is *connected* if it is not the union of two open non-void disjoint sets. In a connected space the empty set and the whole space are the only sets which are simultaneously open and closed. A subset X of $\mathfrak{X}$ is connected if for every partition $X = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$, we have $(A \cap \bar{B}) \cup (\bar{A} \cap B) \neq \emptyset$. A *component* of a set is a connected subset which is not contained in any larger connected subset. Each point of X determines a unique component containing it and X is the union of its components.

A system S of sets $\{A_\alpha\}$ is a *covering* of the set X if each point of X is interior point of at least one set A_α. The set X has the *Heine-Borel property* if every system of open sets $\{G_\alpha\}$ which covers X contains a finite sub-system also covering X. Following S. Lefschetz we shall say:

A topological space or a subset thereof is compact if it has the Heine-Borel property. It is locally compact whenever each point x of the space has a neighborhood $N(x)$ whose closure is compact.

1.3. Hausdorff spaces. Given a space $\mathfrak{X}$ and a collection of subsets $\{N_\alpha\}$, called *neighborhoods*, the space will be called a *Hausdorff space* or a *T_2-space* if:

H_1. *To every point x there is at least one neighborhood $N(x)$ containing it.*

H_2. *If $N_1(x)$ and $N_2(x)$ are neighborhoods of x, there is at least one neighborhood $N_3(x)$ of x such that $N_3(x) \subset N_1(x) \cap N_2(x)$.*

H_3. *If $N(x)$ is a neighborhood of x, and $y \in N(x)$, then there is a neighborhood $N(y)$ of y with $N(y) \subset N(x)$.*

H_4. *If $x \neq y$, there exist neighborhoods $N(x)$ of x and $N(y)$ of y such that $N(x) \cap N(y) = \emptyset$.*

A Hausdorff space is a topological space provided open and closed sets are defined in an obvious manner in terms of the given neighborhoods, the closure of X being then the intersection of all closed sets containing X. On the other hand, a topological space satisfies conditions H_1–H_3 but ordinarily not the *separation axiom* H_4.

A subset $\{N_\beta\}$ of the given neighborhoods is a *neighborhood base* at the point x if $x \in N_\beta$ for all β and if to every given open set G containing x there is an N_β such that $N_\beta \subset G$. The set $\{N_\beta\}$ is a neighborhood base for the whole space $\mathfrak{X}$ if it contains a base for each point of $\mathfrak{X}$. The case in which there is a countable base is particularly important. The two well known axioms of Hausdorff refer to this situation:

First Countability Axiom. *There is a countable base at each point of the space.*

Second Countability Axiom. *The space has a countable base.*

A Hausdorff space satisfying the second countability axiom is separable.

In a Hausdorff space a sequence $\{x_n\}$ converges to a limit x_0 if and only if for every neighborhood $N(x_0)$ of x_0 there is an m with $x_n \in N(x_0)$ for $n > m$. The limit is unique.

If $\mathfrak{X}_1, \mathfrak{X}_2, \cdots, \mathfrak{X}_n$ are topological spaces, the *product set* $X_1 \times X_2 \times \cdots \times X_n$, $X_1 \subset \mathfrak{X}_1, \cdots, X_n \subset \mathfrak{X}_n$, is the set of all ordered n-tuples $(x_1, x_2, \cdots, x_n)$ where $x_1 \in X_1, \cdots, x_n \in X_n$. In particular, the symbol $\mathfrak{X}_1 \times \mathfrak{X}_2 \times \cdots \times \mathfrak{X}_n$ denotes the *product space* of all such ordered n-tuples. The product space can be made into a topological space by the convention that the point $x = (x_1, \cdots, x_n)$ belongs to the closure of the set X if and only if for every choice of open sets $G_1, \cdots, G_n$ with $x_1 \in G_1, \cdots, x_n \in G_n$ we have $(G_1 \times \cdots \times G_n) \cap X \neq \emptyset$. In particular, the closure of a product set is the product set of the closures. If the given spaces are Hausdorff spaces, we make the product space into a Hausdorff space by defining $\{N_{\alpha_1} \times \cdots \times N_{\alpha_n}\}$ as the system of neighborhoods for the product space where $\{N_{\alpha_i}\}$ is the given system of neighborhoods of the space $\mathfrak{X}_i$.

1.4. Metric spaces. $\mathfrak{X}$ is a *metric space* if for each pair of points x, y in $\mathfrak{X}$ there is defined a real-valued function $d(x, y)$, called the *distance* from x to y, subject to the two postulates of Lindenbaum:

D_1. $d(x, y) = 0$ *if and only if* $x = y$.

D_2. $d(x, y) \leqq d(z, x) + d(z, y)$ *for any three points x, y, and z.*

These two properties imply:

D_3. $d(x, y) \geqq 0$,

D_4. $d(x, y) = d(y, x)$,

and in view of D_4 the *triangle axiom* D_2 may be written

D_2'. $d(x, y) \leqq d(x, z) + d(z, y)$.

In a metric space the set of points x with $d(x_0, x) < \rho$ is called a *sphere*. A point x_0 is an interior point of the set X if there is a sphere $d(x_0, x) < \rho$ which is in X. The subset of interior points of X is Int (X) and X is open if Int $(X) = X$. The point x_0 is a limit point of X if every sphere $d(x_0, x) < \rho$ contains at least two points of X. The set X is closed if it contains its limit points. It is *bounded* if it is contained in a sphere. The *diameter* of X is sup $d(x, y)$ for $x, y \in X$.

A topological space is said to be *normal* if to every pair of disjoint closed sets F_1 and F_2 there are disjoint neighborhoods: $F_1 \subset G_1$, $F_2 \subset G_2$, $G_1 \cap G_2 = \emptyset$. A metric space in which neighborhoods are defined to be the open sets of the space thereby becomes a normal Hausdorff space satisfying the first countability axiom. It is separable if and only if it satisfies the second axiom of countability. According to Urysohn, a topological space which is normal and satisfies the second countability axiom is *metrizable*, that is, a metric topology may be introduced which is equivalent to the original topology so that the property of being closed means the same thing in both topologies.

In a metric space a sequence $\{x_n\}$ converges to the limit x_0 if and only if $d(x_0, x_n) \to 0$. The Cauchy condition: *to every* $\epsilon > 0$ *there is a* $k(\epsilon)$ *such that* $d(x_m, x_n) < \epsilon$ *for* $m, n > k(\epsilon)$, is necessary for convergence. A sequence satisfying this condition is called a *Cauchy sequence* and a metric space is said to be *complete* if and only if every Cauchy sequence converges to a limit. If a set X is of the first category in a complete metric space, then $\bar{X}$ is dense in the space. *A complete metric space is of the second category and cannot be of the first category at any point.*

BAIRE'S CATEGORY THEOREM. *If a complete metric space is represented as the union of a countable set of subsets,* $\mathfrak{X} = \bigcup_n X_n$, *then the closure of* X_n *contains a sphere for at least one value of* n.

A set X in a metric space is *totally bounded* if for every $\delta > 0$ the set may be represented as the union of a finite number of sets each of diameter $< \delta$. A totally bounded set is separable. In a complete metric space the closure of a totally bounded set is compact. Conversely, a compact subset of a metric space is totally bounded and a compact metric space is complete. A metric space is *sequentially compact* if every sequence $\{x_n\}$ contains a subsequence converging to a point in $\mathfrak{X}$. *A sequentially compact metric space is compact and vice versa.*

1.5. Partial ordering. A set X is *partially ordered* if for some pairs of elements x, y an *ordering relation* $x < y$ exists (also denoted by $y > x$) which is *transitive*: $x < y$ and $y < z$ implies $x < z$. The ordering is *reflexive* if $x < x$

for all x, *proper* if $x < y$ together with $y < x$ implies $x = y$. The set X is *simply ordered* if all pairs x, y are ordered.

In a partially ordered system X the subset Y is said to have y_0 for an *upper bound* whenever $y < y_0$ for every $y \in Y$. The element x_0 is *maximal* for X if $x > x_0$ implies $x_0 > x$. For the existence of maximal elements we have the convenient *maximal principle* of Max Zorn, one form of which is

ZORN'S LEMMA. *If X is a partially ordered system in which each simply ordered subset has an upper bound in the system, then X contains at least one maximal element.*

References. Fréchet [4], Kuratowski [2], Lefschetz [1].

2. Additive Spaces

1.6. Algebraic systems. The topological spaces which are most useful in functional analysis are at the same time *algebraic systems*. By this we mean that one or more algebraic operations are defined in the space which is closed under the operations in question. There are essentially three *binary operations*, conventionally referred to as *addition*, *multiplication*, and *scalar multiplication*, which come into consideration in this connection.

The simplest and most primitive systems are obtained when there is only one operation defined, either addition or multiplication. The system is then a *semi-group* or possibly a *group* depending upon the stringency of the postulates. In the case of two operations, there are two useful alternatives: addition and multiplication give rise to a *ring*, addition and scalar multiplication to a *linear system*. Finally if all three operations are defined, the system is an *algebra*. Formal definitions will be given when needed, here only an orientation is desired.

The systems discussed in the remainder of this chapter are determined by three sets of postulates: the first defining the algebraic operations which are permissible, the second the topology of the space, and the third the relations between operations and topology. The purpose of the third set is to ensure continuity of the operations in the particular topology under consideration.

From this enumeration the reader will perceive that there are five basic possibilities of constructing algebraico-topological systems. We shall, however, omit multiplicative spaces as well as ring spaces. The three remaining alternatives, additive, linear, and algebraic spaces, will be highly specialized since we shall have no use for the general theory of such spaces.

1.7. Additive groups. Let $\mathfrak{X}$ be a set, containing at least two distinct elements, in which a binary operation called *addition* is defined.

DEFINITION 1.7.1. *$\mathfrak{X}$ is an additive group if the following conditions hold:*

A_1. *Every ordered pair of elements x, y has a uniquely defined sum $x + y$.*

A_2. *Addition is associative, that is, $(x + y) + z = x + (y + z)$.*

A_3. *There is a zero element θ such that $x + \theta = \theta + x = x$ for all x.*

A_4. *To every x there is a negative, written $-x$, such that $x + (-x) = \theta$.*

Addition is not assumed to be *commutative* for the present. The postulates imply that the zero element and the negative are unique and that $(-x) + x = \theta$ for all x whence it follows that $-(-x) = x$. The properties of the negative give the *law of cancellation*:

$$x + z = y + z \quad \text{or} \quad z + x = z + y \quad \text{implies} \quad x = y.$$

A subset X of $\mathfrak{X}$ is called a *subgroup* if X is an additive group under the same operation.

If X and Y are subsets of $\mathfrak{X}$, the symbols $-X$, $X + Y$, and $X - Y$ denote the sets $\{-x\}$, $\{x + y\}$, and $\{x - y\}$ respectively where $x \in X$, $y \in Y$. In particular $X + y = \{x + y\}$, $y + X = \{y + x\}$ where $x \in X$.

1.8. Additive group spaces. Let $\mathfrak{X}$ be an additive group and suppose that a topology has been introduced in $\mathfrak{X}$ so that $\mathfrak{X}$ becomes a topological space, which, in particular, may be a Hausdorff space or a metric space.

DEFINITION 1.8.1. *An additive group $\mathfrak{X}$ is:*

(1) *a topological additive group if $\mathfrak{X}$ is a topological space and*

AT. $\overline{X - Y} = \bar{X} - \bar{Y}$ *whenever one of the sets X or Y reduces to a single point, the other being arbitrary;*

(2) *a Hausdorff additive group if $\mathfrak{X}$ is a Hausdorff space and*

AH. *to every pair of elements x, y and every neighborhood $N(x - y)$ of $x - y$ there are neighborhoods $N(x)$ of x and $N(y)$ of y such that $N(x) - N(y) \subset N(x - y)$;*

(3) *a metric additive group if $\mathfrak{X}$ is a metric space and*

AD. *whenever $d(x_n, x_0) \to 0$ and $d(y_n, y_0) \to 0$, then $d(x_n - y_n, x_0 - y_0) \to 0$.*

There are a number of definitions of topological groups, additive or multiplicative, in the literature corresponding to different choices of the basic topology and of the continuity postulate. Ordinarily it is required that $x + y$ be a continuous function of (x, y). This is guaranteed by postulates AH and AD above but not by AT. The latter, however, suffices to make the group operations continuous, that is, reflection in the origin and right and left translations are continuous operations and even homeomorphisms under AT. This postulate of course amounts to assuming that the closure operation commutes with the group operations. This in turn implies that all topological properties of subsets of $\mathfrak{X}$ are left invariant by the group operations. Thus if a set X has a certain topological property P, then the sets $X + a$, $a + X$, and $-X$ have the same property for every a.

A similar remark applies to conditions ST and MT of Definitions 1.10.1 and 1.13.1 below. These postulates are sufficient to make scalar multiplication and ring multiplication continuous with respect to each variable separately but not necessarily with respect to both variables simultaneously. For a verification of the continuity properties, see section 2.4 below.

The following theorem on subgroups is due to C. Kuratowski ([2, p. 75] and [1, p. 38]). The proof can serve as a preparation for Theorem 7.7.1.

THEOREM 1.8.1. *If a subgroup G of a topological additive group $\mathfrak{X}$ has the property of Baire, then either G is of the first category in $\mathfrak{X}$ or G is both open and closed in $\mathfrak{X}$ so that $G = \mathfrak{X}$ if $\mathfrak{X}$ is connected.*

PROOF. Suppose that G is of the second category in $\mathfrak{X}$ and form the set $D(G)$ which is not empty. Since $D(G)$ is the closure of its own interior, Int $[D(G)] \neq \emptyset$. G being of the second category at every point of the open set Int $[D(G)]$, we conclude that the intersection of G and Int $[D(G)]$ is not empty.

Let us now consider how these sets are transformed by the operations of G. If $g \in G$, then $G + g = G$ and $-G = G$. We have also $g + G = G$, but we shall not need the use of left translations in the following. Since X and $X + g$ have the same topologial properties, Int $(X + g) =$ Int $(X) + g$. Thus

$$\text{Int}\,[D(G)] = \text{Int}\,[D(-G)] = \text{Int}\,[-D(G)] = -\text{Int}\,[D(G)],$$

$$\text{Int}\,[D(G)] = \text{Int}\,[D(G + g)] = \text{Int}\,[D(G) + g] = \text{Int}\,[D(G)] + g,$$

or Int $[D(G)]$ and, a fortiori, its closure $D(G)$ are invariant under the operations of G. But this implies that if one point of G belongs to Int $[D(G)]$, so do all points of G. Hence

$$G \subset \text{Int}\,[D(G)] \subset D(G) \subset \bar{G}.$$

If $p \in \bar{G}$, then every neighborhood of p contains elements of G. Let g be a point of G in the open set Int $[D(G)] + p$ which is clearly a neighborhood of p. Then $g - p \in$ Int $[D(G)]$. Hence also $p - g \in$ Int $[D(G)]$ and finally $p \in$ Int $[D(G)]$. In other words, $\bar{G} \subset$ Int $[D(G)]$ and consequently

$$\text{Int}\,[D(G)] = \bar{G}.$$

The set on the left is open, the one on the right is closed. Hence $\bar{G}$ is both open and closed. If $\mathfrak{X}$ is supposed to be connected, we conclude that $\bar{G} = \mathfrak{X}$.

So far we have not used the assumption that G has the property of Baire. Since $\bar{G} = D(G)$ is an open set and G is of the second category at all of its points, it follows that $\tilde{G}$ is of the first category at some point of $\bar{G}$. Suppose that p is a point of $\bar{G}$ not in G and form the coset $G + p$. It has no points in common with G, that is, $G + p \subset \tilde{G}$; it is of the second category and has the property of Baire since topological properties of G are unchanged by right translations. It follows that $G + p$ is of the second category at all points of its own closure which, however, equals $\bar{G} + p$. But $\bar{G}$ is also a subgroup of $\mathfrak{X}$ so that $\bar{G} + p = \bar{G}$ and $G + p$ is of the second category at all points of $\bar{G}$. It follows that $\tilde{G}$, of which $G + p$ is a subset, is also of the second category at all points of $\bar{G}$. The contradiction implies that $\bar{G} = G$, or G is both open and closed and equals $\mathfrak{X}$ if $\mathfrak{X}$ is connected. This completes the proof.

References. Alexandroff-Hopf [1], Banach [2], van Dantzig [1], Hille [8], Kuratowski [1, 2], Lefschetz [1], Pontrjagin [1], A. Weil [1, 2].

3. Linear Spaces

1.9. Linear systems. In an additive group $\mathfrak{X}$ we introduce a second operation, that of *scalar multiplication*. The scalars could form any *domain of integrity* without essential modification of the subsequent postulates, but for the sake of simplicity we restrict ourselves to the most important case in which the scalars are either *real or complex numbers*. The scalar field will be denoted by Φ.

Definition 1.9.1. *$\mathfrak{X}$ is a linear system* (= module over the scalar field Φ) *if its elements admit of the two operations of addition and scalar multiplication, subject to the following conditions.*

Addition satisfies postulates A_1 *to* A_4 *of Definition* 1.7.1 *together with*

A_5. *Addition is commutative*: $x + y = y + x$.

Scalar multiplication satisfies:

S_1. *To every number $\alpha \in \Phi$ and every element $x \in \mathfrak{X}$, there is a uniquely defined scalar product $\alpha x = x\alpha$ in $\mathfrak{X}$.*

S_2. $(\alpha + \beta)x = \alpha x + \beta x$.

S_3. $\alpha(x + y) = \alpha x + \alpha y$.

S_4. $\alpha(\beta x) = \alpha\beta x$.

S_5. $1 \cdot x = x$.

From S_2, S_5, and A_3 it follows that $-x = (-1)x$ and that $0 \cdot x = \theta$ for all x.

If X is a subset of $\mathfrak{X}$ and A a subset of Φ, the symbol AX stands for the set $\{\alpha x\}$ where $\alpha \in A$, $x \in X$.

A subset X of $\mathfrak{X}$ is said to be *convex* if $x_1, x_2 \in X$ implies that $\alpha x_1 + (1 - \alpha)x_2 \in X$ for $0 < \alpha < 1$.

A linear system, with or without an imposed topology, is usually referred to as a *linear vector space* and the elements are called *vectors*. A system of n vectors $x_1, x_2, \cdots, x_n$, none of which is zero, is said to be *linearly independent* if the equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \theta, \qquad \alpha_k \in \Phi,$$

implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. They are *linearly dependent* if such an equation holds in which at least one coefficient is different from zero. If $\mathfrak{X}$ contains n linearly independent vectors, but every system of $(n + 1)$ vectors is linearly dependent, then $\mathfrak{X}$ is said to be of *dimension n*. If there is no limit to the number of independent vectors, then $\mathfrak{X}$ is said to be of *infinite dimension*.

1.10. Linear spaces. We suppose that $\mathfrak{X}$ is a linear system and proceed to introduce a topology in $\mathfrak{X}$ in such a manner that the arithmetical operations become continuous.

Definition 1.10.1. *A linear system $\mathfrak{X}$ is*:

(1) *a topological linear space if $\mathfrak{X}$ is a topological additive group and*

ST. $\overline{AX} = \bar{A}\bar{X}$ *whenever one of the sets A or X reduces to a single point, the other being arbitrary*;

(2) *a Hausdorff linear space if* $\mathfrak{X}$ *is a Hausdorff additive group and*

SH. *to every* $\alpha \in \Phi$, $x \in \mathfrak{X}$ *and every neighborhood* $N(\alpha x)$ *of* αx *there are neighborhoods* $N(\alpha)$ *of* α *and* $N(x)$ *of* x *such that* $N(\alpha)N(x) \subset N(\alpha x)$;

(3) *a metric linear space if* $\mathfrak{X}$ *is a metric additive group and*

SD. *whenever* $\alpha_n \to \alpha_0$ *and* $d(x_n, x_0) \to 0$, *then* $d(\alpha_n x_n, \alpha_0 x_0) \to 0$.

Assumptions AT and ST of course imply that the closure operation commutes with the arithmetical operations of the linear system, that is, with addition and scalar multiplication, so that all topological properties of subsets of the space remain invariant under the arithmetical operations. Thus if a set X has a topological property P, so do the sets $X + y$ and αX ($\alpha \neq 0$). The same is true in the Hausdorff and the metric cases. The metric linear spaces contain the (F)-spaces of Banach as a special instance.

If $\mathfrak{L} \subset \mathfrak{X}$ and $\mathfrak{L}$ is a linear space, then $\mathfrak{L}$ is called a *linear subspace* of $\mathfrak{X}$ or a *linear manifold* in $\mathfrak{X}$. To each set X in $\mathfrak{X}$ there is a *least linear subspace* (X) which contains X. It is made up of all elements of the form $\sum_{k=1}^{n} \alpha_k x_k$ where n is finite, $\alpha_k \in \Phi$, and $x_k \in X$. A closed linear subspace is a linear subspace which is closed in $\mathfrak{X}$. The closure of (X) is the *least closed linear subspace* containing X.

The analog of Theorem 1.8.1 for linear spaces reads:

Theorem 1.10.1. *If a linear subspace of a topological linear space has the property of Baire, then it is either of the first category or is the whole space.*

For a proof, see E. Hille [8, p. 380]. Note that a topological linear space is connected. The theorem is true for more general scalar systems but presupposes that Φ is connected.

1.11. Banach spaces. By far the most important class of metric linear spaces are the *Banach spaces*, (B)-spaces for short. Here the metric topology is based upon a *norm*.

Definition 1.11.1. *A linear system* $\mathfrak{X}$ *is called a* (B)*-space if*

(1) *with every element* x *there is associated a real number* $\| x \|$, *called the norm of* x, *with the properties*

N_1. $\| x \| \geqq 0$ *and* $\| x \| = 0$ *if and only if* $x = \theta$,

N_2. $\| \alpha x \| = | \alpha | \, \| x \|$,

N_3. $\| x + y \| \leqq \| x \| + \| y \|$;

(2) $d(x, y) = \| x - y \|$;

(3) $\mathfrak{X}$ *is complete in the resulting topology.*

$\mathfrak{X}$ *is a real or a complex* (B)*-space according as* Φ *is the real or the complex number field.*

Verification of the fact that a (B)-space is a metric linear space in the sense of Definition 1.10.1 is left to the reader. Numerous examples of special (B)-spaces are given in Banach's treatise [2]; other examples are to be found in Part III of this book.

In any (B)-space the interior of the unit sphere is a convex set. J. A. Clarkson [1] has introduced (B)-spaces having a stronger convexity property:

A (B)-space is said to be uniformly convex if $\| x_n \| = 1, \| y_n \| = 1, n = 1, 2, 3, \cdots$, *together with* $\lim_{n\to\infty} \| \frac{1}{2}(x_n + y_n) \| = 1$ *implies* $\lim_{n\to\infty} \| x_n - y_n \| = 0$.

The Lebesgue spaces $L_p(a, b)$ with $p > 1$ are examples of uniformly convex spaces. On the other hand, $L_1(a, b)$ and $C[a, b]$ are not uniformly convex.

Each n-dimensional euclidean space with the usual metric is a locally compact (B)-space. Conversely, by a theorem of S. Banach [2, p. 84] which goes back to F. Riesz:

A locally compact (B)-space is of finite dimension, that is, there is a finite set of elements $u_1, u_2, \cdots, u_n$ such that every element of $\mathfrak{X}$ is of the form $x = \alpha_1 u_1 + \cdots + \alpha_n u_n$ with coefficients in Φ.

A countable set of elements $\{u_n\}$ is said to be a *base* of the (B)-space $\mathfrak{X}$ if for each $x \in \mathfrak{X}$ there is one and only one sequence of numbers $\{\alpha_n\}$ in Φ such that $x = \sum_{n=1}^{\infty} \alpha_n u_n$.

References. Banach [2], Clarkson [1], Hille [8], F. Riesz [3].

4. Algebraic Spaces

1.12. Algebras. An algebraic system in which all three operations are defined is known as an *algebra*.

Definition 1.12.1. *$\mathfrak{A}$ is an algebra over the scalar field Φ if its elements admit of the three operations of addition, multiplication, and scalar multiplication, subject to the following conditions.*

$\mathfrak{A}$ is a linear system in the sense of Definition 1.9.1.

The multiplication satisfies:

M_1. *Every ordered pair of elements x, y has a unique product xy.*

M_2. *Multiplication is associative*: $(xy)z = x(yz)$.

D. *Addition and multiplication are distributive*:

$$x(y + z) = xy + xz, \quad (y + z)x = yx + zx.$$

S_6. *Multiplication and scalar multiplication commute*: $\alpha x \beta y = \alpha\beta xy$.

Further conditions which will sometimes be imposed are:

M_3. *There exists a unit element e such that $ex = xe = x$ for each x.*

M_4. *Multiplication is commutative*: $xy = yx$.

We speak of an *algebra with a unit element* if M_3 holds, a *commutative algebra* if M_4 holds.

If X and Y are two given subsets of $\mathfrak{A}$, the symbol XY will denote the set $\{xy\}$, $x \in X$, $y \in Y$.

1.13. Algebraic spaces. Topologies are introduced as in the previous cases.

DEFINITION 1.13.1. *An algebra* $\mathfrak{A}$ *is*

(1) *a topological algebra* (= *topological algebraic space*) *if* $\mathfrak{A}$ *is a topological linear space and*

MT. $\overline{XY} = \bar{X}\bar{Y}$ *whenever one of the sets* X *or* Y *reduces to a single point, the other set being arbitrary;*

(2) *a Hausdorff algebra if* $\mathfrak{A}$ *is a Hausdorff linear space and*

MH. *to every neighborhood* $N(xy)$ *of* xy *there are neighborhoods* $N(x)$ *of* x *and* $N(y)$ *of* y *such that* $N(x)N(y) \subset N(xy)$;

(3) *a metric algebra if* $\mathfrak{A}$ *is a metric linear space and*

MD. *whenever* $d(x_n, x_0) \to 0$, $d(y_n, y_0) \to 0$, *then* $d(x_ny_n, x_0y_0) \to 0$.

In all three cases the arithmetical operations of the algebra commute with the closure operation, so that if X has a certain topological property, the sets $X + y$, αX $(\alpha \neq 0)$, yX, and Xy have the same property.

If $\mathfrak{A}_1 \subset \mathfrak{A}$ and $\mathfrak{A}_1$ is an algebra, then $\mathfrak{A}_1$ is a *subalgebra* of $\mathfrak{A}$. To every subset X of $\mathfrak{A}$ there exists a *least subalgebra* $\mathfrak{A}(X)$ containing X. It consists of all finite multinomials obtained by adding and multiplying elements of X, the coefficients being in Φ. The closure of $\mathfrak{A}(X)$ is the *least closed subalgebra* containing X. *A proper subalgebra which has the property of Baire must be of the first category in* $\mathfrak{A}$.

1.14. Banach algebras. By far the most important instance of a metric algebra is given by the *Banach algebras* which are usually known under the name of *normed vector rings*. The latter name being a misnomer from the algebraic standpoint, we have adopted the term Banach algebra, (B)-algebra for short, following a suggestion of Max Zorn.

DEFINITION 1.14.1. $\mathfrak{B}$ *is a Banach algebra if* $\mathfrak{B}$ *is an algebra as well as a Banach space and if, in addition,*

$$\| xy \| \leqq \| x \| \, \| y \|.$$

It is a real or a complex Banach algebra according as Φ *is the real or the complex number field.*

The inequality

$$\begin{aligned} \| xy - x_ny_n \| &\leqq \| x(y - y_n) \| + \| (x - x_n)y_n \| \\ &\leqq \| x \| \, \| y - y_n \| + \| y_n \| \, \| x - x_n \| \end{aligned}$$

shows that xy is a continuous function of both variables together.

If $\mathfrak{B}$ has a unit element e, we have $\| e \| \geqq 1$. I. Gelfand has shown that it is always possible to assume that $\| e \| = 1$. If this is not true at the start, we can always replace $\mathfrak{B}$ by a (B)-algebra $\mathfrak{B}'$ which is algebraically as well as topologically equivalent to $\mathfrak{B}$ ($\mathfrak{B}'$ is an isomorphic and homeomorphic image of $\mathfrak{B}$) whose unit element has the norm one.

References. Ambrose [1], van Dantzig [1], Gelfand [4].

CHAPTER II

LINEAR OPERATIONS

2.1. Orientation. In this chapter we shall give a survey of those parts of *the theory of linear operations* which have a bearing on our main problem. More proofs are included in the text than in the preceding chapter, but the presentation is far from self-contained and the References are recommended for supplementary reading. The material is grouped under five paragraph headings: *Continuous Transformations*, *Linear Transformations*, *Linear Functionals*, *Linear Bounded Transformations*, and *Spaces of Endomorphisms*.

1. Continuous Transformations

2.2. Mappings. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two topological spaces, in the sense of section 1.2, which may be identical or distinct. Let $y = T(x)$ be a single-valued function with *domain* $\mathfrak{D}$ in $\mathfrak{X}$ and *range* $\mathfrak{R}$ in $\mathfrak{Y}$, that is, to every x in $\mathfrak{D}$ corresponds a unique y in $\mathfrak{R}$ and every y in $\mathfrak{R}$ is the image of at least one x in $\mathfrak{D}$. Then $y = T(x)$ is said to define a *mapping of* $\mathfrak{D}$ *onto* $\mathfrak{R}$ or a *transformation on* $\mathfrak{D}$ *to* $\mathfrak{R}$.

If $X \subset \mathfrak{D}$, the symbol $T(X)$ denotes the set of all points $T(x)$ with $x \in X$ and $T(X)$ is called the *image* of X under the mapping. If $Y \subset \mathfrak{R}$ then the set of all points x in $\mathfrak{D}$ with $T(x) \in Y$ is called the *inverse image* of Y and is denoted by $T^{-1}(Y)$. It is clear that

$$Y = T[T^{-1}(Y)] \text{ for all } Y \subset \mathfrak{R},$$

$$X \subseteq T^{-1}[T(X)] \text{ for all } X \subset \mathfrak{D},$$

and, in general, the inclusion is proper. The mapping is one-to-one if equality holds, that is, if $Y = T(X)$ implies $X = T^{-1}(Y)$ for every $X \subset \mathfrak{D}$. In this case there exists a unique single-valued function $x = T^{-1}(y)$ with domain $\mathfrak{R}$ and range $\mathfrak{D}$ such that

$$T[T^{-1}(y)] = y \text{ for all } y \in \mathfrak{R} \text{ and } T^{-1}[T(x)] = x \text{ for all } x \in \mathfrak{D}.$$

We call $x = T^{-1}(y)$ the *inverse* of $y = T(x)$. The point set $[x, T(x)]$, $x \in \mathfrak{D}$, in the product space $\mathfrak{X} \times \mathfrak{Y}$ is known as the *graph* of $T(x)$.

2.3. Continuity. For the sake of simplicity we take $\mathfrak{D} = \mathfrak{X}$ and denote closure by the same symbol in both spaces.

DEFINITION 2.3.1. *The transformation $y = T(x)$ is continuous at $x = x_0$ if for every set X the condition $x_0 \in \bar{X}$ implies that $T(x_0) \in \overline{T(X)}$. $T(x)$ is a continuous transformation (mapping) if $T(x)$ is continuous at all points.*

THEOREM 2.3.1. *Necessary and sufficient conditions in order that $T(x)$ be a continuous transformation are*

(1) *$T(\bar{X}) \subset \overline{T(X)}$ for every set X, or*

(2) *$T^{-1}(G)$ is open for every open set G, or*

(3) *$T^{-1}(F)$ is closed for every closed set F.*

For a proof see C. Kuratowski [**2**, pp. 67–68]. Functions of two or more variables are discussed in the same manner by introducing the product space on which they are defined. In the case of a Hausdorff space satisfying the first countability axiom and, in particular, if the space is metric, the conditions may be given the classical form: *$T(x)$ is continuous at $x = x_0$ if and only if $x_n \to x_0$ implies that $T(x_n) \to T(x_0)$.*

A continuous transformation maps compact sets onto compact sets.

DEFINITION 2.3.2. *If $y = T(x)$ has a unique inverse and is continuous together with its inverse, then $T(x)$ is called a topological transformation and the mapping is a homeomorphism.*

If $\mathfrak{X}$ and $\mathfrak{Y}$ are complete metric spaces, if $\{T_n(x)\}$, $n = 1, 2, 3, \cdots$, are transformations with domain $\mathfrak{D}_n \subset \mathfrak{X}$ and range $\mathfrak{R}_n \subset \mathfrak{Y}$, and if $\lim_{n\to\infty} T_n(x)$ exists for each x in $\mathfrak{D} \subset \bigcap_n \mathfrak{D}_n$, then the limit defines a transformation $T(x)$ with domain $\mathfrak{D}$ and the sequence $\{T_n\}$ is said to converge to T in $\mathfrak{D}$.

DEFINITION 2.3.3. *If $\mathfrak{X}$ and $\mathfrak{Y}$ are complete metric spaces, and if $y = T(x)$ has $\mathfrak{X}$ as its domain and has its range in $\mathfrak{Y}$, and if $T(x)$ takes bounded sets in $\mathfrak{X}$ into totally bounded sets in $\mathfrak{Y}$, then $T(x)$ is called a compact transformation.*

Here the usual term is "completely continuous transformation." This is a misnomer, however, since a compact transformation need not be continuous anywhere. Thus the function $f(x)$ which is zero or one according as x is rational or irrational defines a compact transformation on reals to reals which is nowhere continuous.

2.4. Continuity of the arithmetical operations. In the definitions of sections 1.8, 1.10, and 1.13 conditions were laid down with the stated purpose of making the arithmetical operations continuous in additive groups, linear spaces, and algebraic spaces, but no attempt was made to verify the continuity. Theorem 2.3.1 (1) provides a tool for such a proof. We shall carry through the argument for additive groups, mainly as an illustration of the theorem.

Applying condition (1) to the transformation $y = -x$ we see that $-\bar{X} \subset \overline{-X}$ for all sets X is necessary and sufficient for continuity. Replacing X by $-X$ reverses the inclusion and gives the final condition $\overline{-X} = -\bar{X}$ for all X. The transformation $y = x + a$ is continuous in x if and only if $\bar{X} + a \subset \overline{X + a}$ is true for all X. If this holds for all a, then all right translations will be continuous together with their inverses, that is, they become homeomorphisms. This implies that $\overline{X + a} - a \subset \overline{X + a - a} = \bar{X}$ whence $\overline{X + a} = \bar{X} + a$ for all X and all a. Similarly the left translations become homeomorphisms if and only if

$\overline{a + X} = a + \bar{X}$ for all X and all a. Combining all three conditions, we see that Postulate AT results. This postulate is consequently equivalent to the requirement that the three operations of reflection in the origin, left translations, and right translations be homeomorphisms.

The same type of argument shows that Postulate ST is necessary and sufficient in order that scalar multiplication be continuous with respect to each variable separately. Similarly, Postulate MT is necessary and sufficient for the continuity of ring multiplication with respect to each of the factors.

References. Kuratowski [2], Lefschetz [1].

2. Linear Transformations

2.5. Additive transformations. In the following both domain and range spaces will be algebraic systems. For the sake of simplicity we use the same symbols for corresponding operations in the two spaces and do not distinguish between the two zero elements.

DEFINITION 2.5.1. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two topological additive groups and $y = T(x)$ a transformation on $\mathfrak{X}$ into $\mathfrak{Y}$. $T(x)$ is said to be additive if for all x_1 and x_2 we have*

$$T(x_1 + x_2) = T(x_1) + T(x_2).$$

The definition implies that

$$T(\theta) = \theta, \qquad T(-x) = -T(x).$$

THEOREM 2.5.1. *An additive transformation is continuous everywhere if it is continuous at a single point.*

PROOF. Suppose that $T(x)$ is continuous at $x = x_0$, take $x_1 \neq x_0$, and let X be any set with $x_1 \in \bar{X}$. Then

$$x_0 \in \bar{X} - x_1 + x_0 = \overline{X - x_1 + x_0}$$

and by Definition 2.3.1 this implies

$$T(x_0) \in \overline{T(X - x_1 + x_0)} = \overline{T(X) - T(x_1) + T(x_0)} = \overline{T(X)} - T(x_1) + T(x_0)$$

whence $T(x_1) \in \overline{T(X)}$. Hence $T(x)$ is continuous at the arbitrary point x_1.

2.6. Linear transformations. We turn now to linear spaces. We recall that the scalar field Φ is either the real or the complex field.

DEFINITION 2.6.1. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two topological linear spaces with the same scalar field Φ and let $y = T(x)$ be a transformation on $\mathfrak{X}$ into $\mathfrak{Y}$. If $T(\alpha x) =$*

$\alpha T(x)$ for all x and all real α, then $T(x)$ is said to be real-homogeneous; $T(x)$ is homogeneous if the relation holds for all $\alpha \in \Phi$.

The two notions obviously coincide if Φ is the real field.

DEFINITION 2.6.2. *An additive and homogeneous transformation is said to be linear.*

The reader should observe that we are not following the usage of the Polish school according to which a linear transformation is also continuous. The various properties are not completely independent, however, as is shown by

THEOREM 2.6.1. *An additive continuous transformation on one topological linear space to another with the same scalar field is linear if either* (i) *Φ is the real field or* (ii) *Φ is the complex field and, in addition, $T(ix) = iT(x)$ for all x.*

PROOF. The additivity obviously implies $T(\rho x) = \rho T(x)$ for each rational ρ. Let P be the rational subfield, let α be irrational, and A an open set containing α so that α is in the closure of $A \cap$ P. Since $T(x)$ is continuous, $T(\alpha x)$ is in the closure of $T((A \cap \text{P})x) = (A \cap \text{P})T(x)$ and the closure of the right member is the scalar product of the closure of $(A \cap \text{P})$ with $T(x)$. But the closure of $(A \cap \text{P})$ equals $\bar{A}$, so $T(\alpha x) \in \bar{A}T(x)$ for every open set A containing α. But the only point common to all sets $\bar{A}$ with $\alpha \in A$ is α itself. Hence $T(\alpha x) = \alpha T(x)$ and $T(x)$ is real-homogeneous. The rest is trivial.

2.7. Boundedness. In (B)-spaces the notion of boundedness is fundamental in the theory of linear transformations.

DEFINITION 2.7.1. *A transformation $y = T(x)$ on one (B)-space to another is said to be bounded if there exists a fixed non-negative M with*

$$\| T(x) \| \leqq M \| x \|$$

for all x. The least value of M satisfying the inequality is called the bound or the norm of T and is denoted by $\| T \|$.

The reader should observe that on the left side of the inequality figures the norm as defined in the space $\mathfrak{Y}$ while on the right the norm refers to the space $\mathfrak{X}$. As long as there is no danger of confusion, we shall not use separate notation for distinct norms.

THEOREM 2.7.1. *An additive transformation on one (B)-space to another is continuous if and only if it is bounded.*

PROOF. If $T(x)$ is bounded, then

$$\| T(x) - T(x_0) \| = \| T(x - x_0) \| \leqq \| T \| \| x - x_0 \|$$

and continuity follows. On the other hand, if $T(x)$ is not bounded, then it cannot be bounded for $\| x \| \leqq 1$. We can then find a sequence of points $\{x_n\}$ with $\| x_n \| \leqq 1$, $\| T(x_n) \| \geqq n$. Then $y_n = x_n/n$ is a null sequence and $\| T(y_n) \| \geqq 1$ for all n. It follows that $T(x)$ cannot be continuous at $x = \theta$ and hence nowhere.

References. Banach [2], Hille [8].

3. Linear Functionals

2.8. Subadditive functionals. A numerically-valued function on an abstract space $\mathfrak{X}$ is known as a *functional.* In the following we restrict ourselves to the case in which $\mathfrak{X}$ is a (B)-space; $\mathfrak{Y}$ is the real or the complex number field. The notions of additive, homogeneous, linear, and continuous functionals are obtained from the definitions in the preceding sections of this chapter by taking $\mathfrak{X}$ as a (B)-space and $\mathfrak{Y}$ as the real or complex vector space. We need a new notion, that of *subadditivity* which will play some role in the present paragraph and special instances of which will be studied at length in Chapter VI.

Definition 2.8.1. *A real-valued functional $F(x)$ is said to be subadditive if for all x_1 and x_2 in $\mathfrak{X}$ we have*

$$F(x_1 + x_2) \leqq F(x_1) + F(x_2).$$

We shall prove a couple of simple properties of subadditive functionals. We note first that *if a subadditive functional is non-negative outside a sphere $\| x \| = R$, then it is non-negative for all x.* Indeed, if $\| x \| < R$, then there is a positive integer n with $n \| x \| \geqq R$ unless $x = \theta$. Hence $0 \leqq F(nx) \leqq nF(x)$ and $F(x) \geqq 0$. For $x = \theta$ we have $F(\theta) \leqq 2F(\theta)$ so that $F(\theta) \geqq 0$. Here we have tacitly assumed $F(x)$ to be finite-valued, if not we may have $F(\theta) = -\infty$. However, since $F(x) \leqq F(x) + F(\theta)$, the assumption $F(\theta) = -\infty$ would lead to $F(x) \equiv -\infty$ against the assumption.

Theorem 2.8.1. *A subadditive functional which is continuous at $x = \theta$ with $F(\theta) = 0$ is continuous for all x and there exists an M such that $-M \| x \| \leqq F(x) \leqq M[\| x \| + 1]$.*

Proof. The double inequality

$$F(x) - F(-h) \leqq F(x + h) \leqq F(x) + F(h)$$

shows that when $h \to \theta$

$$F(x) \leqq \liminf F(x + h) \leqq \limsup F(x + h) \leqq F(x),$$

so that $F(x)$ is continuous everywhere. Next, there is an M such that $F(x) \leqq M$ for $\| x \| \leqq 1$. Indeed, since $F(x) \leqq nF(x/n)$ for every positive integer n, the contrary assumption would imply that $F(x)$ could take on arbitrarily large positive values in any ϵ-sphere about the origin, which clearly contradicts the continuity of $F(x)$. Further, $0 = F(\theta) \leqq F(x) + F(-x)$ gives $F(x) \geqq -F(-x) \geqq -M$ and $| F(x) | \leqq M$ for $\| x \| \leqq 1$. If $n - 1 \leqq \| x \| < n$, we have $F(x) \leqq nF(x/n) \leqq nM \leqq M[\| x \| + 1]$ and here we may replace $F(x)$ by $|F(x)|$. The lower bound for $F(x)$ may be improved upon, however. For any $\epsilon > 0$ we can find an $R = R_\epsilon$ such that $F(x) \geqq -(M + \epsilon) \| x \|$ when $\| x \| \geqq R_\epsilon$. Now, $F(x) + (M + \epsilon) \| x \|$, being the sum of two subadditive functionals, is also a subadditive functional and, being non-negative for $\| x \| \geqq R_\epsilon$, it has to be non-negative for all x. Since this holds for every $\epsilon > 0$, we have $F(x) \geqq -M \| x \|$ and the theorem is proved.

REMARK. The theorem does not admit of much improvement in any direction. Thus the conclusion does not necessarily hold if $F(x)$ is continuous at $x = \theta$ but $F(\theta) \neq 0$ or if $F(x)$ is continuous for an $x \neq \theta$. For the case in which $\mathfrak{X} = E_1$ (euclidean one-dimensional space), counter examples are furnished in connection with the proof of Theorem 6.8.2 below. Finally, the function $F(x) = \| x \|^\alpha$, $0 < \alpha < 1$, shows that an inequality of the form $F(x) \leqq M \| x \|$ need not hold for all x.

COROLLARY. *In a (B)-space the norm of x is a continuous function of x.*

2.9. Extension of linear functionals. Let $\mathfrak{X}$ be a given (B)-space and consider the linear bounded functionals on $\mathfrak{X}$ which will be denoted by the generic symbol $x^* = x^*(x)$ in the following. We recall that x^* is *linear* if

$$x^*(\alpha x + \beta y) = \alpha x^*(x) + \beta x^*(y) \tag{2.9.1}$$

for all $\alpha, \beta \in \Phi$ and $x, y \in \mathfrak{X}$. It is *bounded*, if for all x

$$| x^*(x) | \leqq M \| x \| \tag{2.9.2}$$

and its *norm* has the value

$$\| x^* \| = \sup | x^*(x) |, \qquad \| x \| \leqq 1. \tag{2.9.3}$$

It is of fundamental importance to know if any linear bounded functionals are available on a given (B)-space and, if so, if there are sufficiently many to distinguish between distinct elements of $\mathfrak{X}$. We have of course always the trivial functional $x^*(x) \equiv 0$, but are there any others? In particular, are there functionals vanishing on a preassigned proper linear subspace of $\mathfrak{X}$ without vanishing identically? To what extent can a linear functional be preassigned?

In principle continuous linear functionals could exist in any linear topological space. However, J. P. LaSalle [1] has shown that *non-zero linear functionals exist if and only if the space contains an open convex set, containing the origin but not the whole space.* On the other hand, according to A. Kolmogoroff [3] the existence of a *bounded* open convex set is necessary and sufficient for the existence of an equivalent normed topology and J. V. Wehausen [1] has shown that the existence of a linear continuous non-zero functional implies a normed topology for equivalence classes modulo the functional in question. Examples of topological spaces in which $x^* \equiv 0$ is the only linear functional are given by the space of measurable functions (S. Banach [2, p. 234]) and the space L_p with $0 < p < 1$ (M. M. Day [1]).

For (B)-spaces the questions raised above may be attacked with the aid of the Hahn-Banach extension theorem in the real case and the Bohnenblust-Sobczyk analog in the complex case. As point of departure one may take the following theorem for which the reader is referred to S. Banach [2, pp. 27–29].

THEOREM 2.9.1. *Given a real (B)-space $\mathfrak{X}$ and*
(1) *a subadditive positive-homogeneous functional $p(x)$ defined on $\mathfrak{X}$:*

$$p(x + y) \leqq p(x) + p(y), \quad p(\alpha x) = \alpha p(x) \text{ for } \alpha > 0;$$

(2) *a real linear functional* $f(x)$ *defined on a linear subspace* $\mathfrak{L}$ *of* $\mathfrak{X}$ *such that* $f(x) \leqq p(x)$, $x \in \mathfrak{L}$.

Then there exists a linear functional $F(x)$ *defined on* $\mathfrak{X}$ *such that* $F(x) \leqq p(x)$ *for all* $x \in \mathfrak{X}$ *and* $F(x) = f(x)$ *in* $\mathfrak{L}$.

It should be observed that the inequality $F(x) \leqq p(x)$ implies $F(-x) \leqq p(-x)$ so that the final conclusion is

$$-p(-x) \leqq F(x) \leqq p(x). \tag{2.9.4}$$

The Hahn-Banach extension theorem is an immediate corollary of Theorem 2.9.1. It reads as follows:

THEOREM 2.9.2. *Given a real (B)-space* $\mathfrak{X}$, *a linear subspace* $\mathfrak{L}$, *and a real linear bounded functional* $f(x)$ *defined on* $\mathfrak{L}$. *Then there exists a real linear bounded functional* $x^*(x)$ *defined on* $\mathfrak{X}$ *such that* $x^*(x) = f(x)$ *on* $\mathfrak{L}$ *and the norm of* $x^*(x)$ *on* $\mathfrak{X}$ *is the same as the norm of* $f(x)$ *on* $\mathfrak{L}$.

PROOF. If

$$M = \| f \|_{\mathfrak{L}} = \sup | f(x) |, \qquad x \in \mathfrak{L}, \ \| x \| \leqq 1,$$

we take $p(x) = M \| x \|$ and apply Theorem 2.9.1. Formula (2.9.4) ensures that the resulting extension is a bounded linear functional.

On the basis of this theorem one may now prove various existence theorems for linear bounded functionals. A couple of results which go back to H. Hahn [2] are listed in the next two theorems.

THEOREM 2.9.3. *To each point* x_0 *of* $\mathfrak{X}$ *there exists a linear bounded functional on* $\mathfrak{X}$ *such that* $x^*(x_0) = \| x_0 \|$ *and* $\| x^* \| = 1$.

PROOF. We may assume $x_0 \neq \theta$. Then the elements of the form αx_0, α real, form a linear subspace $\mathfrak{L}$ on which we define $f(\alpha x_0) = \alpha \| x_0 \|$ so that $\| f \|_{\mathfrak{L}} = 1$. The desired result then follows from the preceding theorem. This shows the existence of infinitely many linear bounded functionals on $\mathfrak{X}$.

THEOREM 2.9.4. *To each given linear subspace* $\mathfrak{L} \subset \mathfrak{X}$ *and each point* x_0 *at a positive distance* d *from* $\mathfrak{L}$ *there is a linear bounded functional* x^* *such that* (i) $x^*(x) = 0$ *on* $\mathfrak{L}$, (ii) $x^*(x_0) = 1$, *and* (iii) $\| x^* \| = 1/d$.

PROOF. If x_1 ranges over $\mathfrak{L}$ and α is real, the elements of the form $x_1 + \alpha x_0$ make up a linear subspace $\mathfrak{L}_1$ of $\mathfrak{X}$. If $x \in \mathfrak{L}_1$, then $x = x_1 + \alpha x_0$ where x_1 and α are uniquely determined since $d \neq 0$. We define $f(x) = \alpha$ on $\mathfrak{L}_1$. It is obvious that $f(x)$ is a linear functional on $\mathfrak{L}_1$ such that $f(x) = 0$ on $\mathfrak{L}$ and $f(x_0) = 1$. It is a simple matter to prove that $\| f \|_{\mathfrak{L}_1} = 1/d$. The extension theorem is then used to complete the proof.

So far $\mathfrak{X}$ was a real (B)-space. Now let $\mathfrak{X}$ be a complex (B)-space instead and $\mathfrak{L}$ a linear subspace of $\mathfrak{X}$. This means that $\mathfrak{L}$ contains all complex linear

combinations of any finite number of its elements. Theorems 2.9.1 and 2.9.2 may still be used to prove the existence of additive, real-homogeneous functionals on $\mathfrak{X}$, but they no longer apply to linear (that is, complex linear) functionals. Here the basic theorem is due to F. Bohnenblust and A. Sobczyk [1]. We obtain their theorem, hereafter referred to as THEOREM 2.9.5, by replacing "real" by "complex" in the wording of Theorem 2.9.2.

For the validity of Theorem 2.9.5 it is essentia that the initial functional $f(x)$, given on the linear subspace $\mathfrak{L}$, is actually linear and not merely additive and real-homogeneous. Indeed, Bohnenblust and Sobczyk have shown that in every complex (B)-space of infinite dimension there exists an additive, real-homogeneous functional defined on a linear subspace which does not admit of a bounded linear extension to the whole space $\mathfrak{X}$.

PROOF OF THEOREM 2.9.5. Let $f(x) = f_1(x) + if_2(x)$ be the linear functional given on $\mathfrak{L}$. Here $f_1(x)$ and $f_2(x)$ are real-valued and a simple calculation shows that they are additive real-homogeneous functionals of norm not exceeding that of $f(x)$ on $\mathfrak{L}$, furthermore $f_2(x) = -f_1(ix)$. As we have already observed, Theorem 2.9.2 may be used to obtain a real additive, real-homogeneous functional $F_1(x)$ on $\mathfrak{X}$ such that $F_1(x) = f_1(x)$ on $\mathfrak{L}$ and $\| F_1 \|_{\mathfrak{X}} = \| f \|_{\mathfrak{L}} = M$. We then set

$$x^*(x) = F_1(x) - iF_1(ix).$$

Since $x^*(ix) = ix^*(x)$, this is a linear bounded functional which obviously coincides with $f(x)$ on $\mathfrak{L}$. If $x^*(x) = re^{i\theta}$ we have

$$| x^*(x) | = e^{-i\theta}x^*(x) = x^*(e^{-i\theta}x) = F_1(e^{-i\theta}x) \leqq M \| x \|.$$

This completes the proof.

We note in passing that Theorems 2.9.3 and 2.9.4 hold for complex (B)-spaces. In constructing the auxiliary functional $f(x)$ we let α take on all complex values instead of real values and then use Theorem 2.9.5 instead of 2.9.2.

The set of points where a given functional x^* vanishes is a closed linear manifold $\mathfrak{X}_0$. If $x^* \not\equiv 0$, $\mathfrak{X}_0 \neq \mathfrak{X}$, and we may find a vector x_1 not in $\mathfrak{X}_0$ such that $x^*(x_1) = 1$. If $x^*(x) = x^*(y)$, then $x - y \in \mathfrak{X}_0$. It follows that with respect to x^* every element x has a unique decomposition, $x = x_0 + \alpha x_1$ where $x_0 \in \mathfrak{X}_0$.

DEFINITION 2.9.1. *A set X_0 in $\mathfrak{X}$ is said to be fundamental if the least closed linear manifold containing X_0 is $\mathfrak{X}$.*

THEOREM 2.9.6. *A necessary and sufficient condition that a set X_0 be fundamental is that every linear bounded functional which vanishes on X_0 vanishes identically.*

PROOF. This is a corollary of Theorem 2.9.4.

THEOREM 2.9.7. *If the dimension of $\mathfrak{X}$ exceeds n and if $x_1^*, \cdots, x_n^*$ are n given bounded linear functionals on $\mathfrak{X}$, then the homogeneous system of equations*

$$x_j^*(x) = 0, \qquad j = 1, 2, \cdots, n,$$

always has non-trivial solutions.

Proof. We may assume the given functionals to be linearly independent and choose $(n + 1)$ linearly independent vectors $x_1, \cdots, x_{n+1}$ in $\mathfrak{X}$ which span a linear subspace $\mathfrak{X}_{n+1}$. If $x \in \mathfrak{X}_{n+1}, x = \xi_1 x_1 + \cdots + \xi_{n+1}x_{n+1}$, and if $x_j^*(x_k) = \alpha_{jk}$ then the system

$$x_j^*(x) \equiv \alpha_{j,1}\xi_1 + \cdots + \alpha_{j,n+1}\xi_{n+1} = 0, \qquad j = 1, 2, \cdots, n,$$

has a non-trivial solution $(\xi_1^0, \cdots, \xi_{n+1}^0)$ which defines a non-trivial solution $x^0 = \xi_1^0 x_1 + \cdots + \xi_{n+1}^0 x_{n+1}$ of the original system.

The problem of solving infinite systems of linear equations in functionals was attacked in special but typical cases by F. Riesz [1, 2] and E. Helly [1] and in the general case by H. Hahn [2]. We state two theorems of Hahn's without proof. The necessity of the stated conditions is obvious and the reader should have no difficulty in deriving the sufficiency from Theorem 2.9.5. The second theorem involves the notion of a reflexive space which is defined in the next section.

Theorem 2.9.8. *Given a complex (B)-space $\mathfrak{X}$, a countable set of elements $\{x_n\}$ in $\mathfrak{X}$, and a countable set of complex numbers $\{C_n\}$. A necessary and sufficient condition for the existence of a linear bounded functional x^* such that* (i) $x^*(x_n) = C_n$ *for all n, and* (ii) $\| x^* \| \leqq M$ *is that the inequality*

$$\left| \sum_1^k \alpha_n C_n \right| \leqq M \left\| \sum_1^k \alpha_n x_n \right\|$$

hold for each k and each choice of the complex numbers $\{\alpha_n\}$.

Theorem 2.9.9. *Given a complex reflexive (B)-space $\mathfrak{X}$, a countable set of linear bounded functionals $\{x_n^*\}$ on $\mathfrak{X}$, and a countable set of complex numbers $\{C_n\}$. A necessary and sufficient condition for the existence of an element x in $\mathfrak{X}$ such that* (i) $x_n^*(x) = C_n$ *for all n and* (ii) $\| x \| \leqq M$ *is that the inequality*

$$\left| \sum_1^k \alpha_n C_n \right| \leqq M \left\| \sum_1^k \alpha_n x_n^* \right\|$$

hold for every integer k and each choice of the complex numbers $\{\alpha_n\}$.

2.10. The adjoint space. Let $\mathfrak{X}$ be a complex (B)-space and consider the set $\mathfrak{X}^*$ of linear bounded functionals on $\mathfrak{X}$. If x_1^* and x_2^* are elements of $\mathfrak{X}^*$ and α_1, α_2 are complex numbers, it is obvious that $\alpha_1 x_1^* + \alpha_2 x_2^*$ is also in $\mathfrak{X}^*$. It follows that addition and scalar multiplication are defined in $\mathfrak{X}^*$ and it is a simple matter of showing that $\mathfrak{X}^*$ is a linear system in the sense of Definition 1.9.1. The zero element of $\mathfrak{X}^*$ is the zero functional which vanishes for all x. We may introduce a normed topology in $\mathfrak{X}^*$ by setting

$$\| x^* \| = \sup | x^*(x) |, \qquad \| x \| \leqq 1.$$

Cf. formula (2.9.3). It is easy to show that this definition gives a proper norm in the sense of Definition 1.11.1 and that $\mathfrak{X}^*$ is complete in the resulting topology. It follows that $\mathfrak{X}^*$ is a complex (B)-space which is known as the *conjugate* or the *adjoint space* of $\mathfrak{X}$. The term *polar space* was used by Hahn to whom we owe much of our knowledge of this field.

The value of the functional $x^* \in \mathfrak{X}^*$ at the point $x \in \mathfrak{X}$ is a complex number $B(x, x^*)$. We may consider $B(x, x^*)$ as a *bilinear functional* since obviously

$$B(\alpha x + \beta y, x^*) = \alpha B(x, x^*) + \beta B(y, x^*), \tag{2.10.1}$$

$$B(x, \gamma x^* + \delta y^*) = \gamma B(x, x^*) + \delta B(x, y^*), \tag{2.10.2}$$

$$| B(x, x^*) | \leqq || x || \, || x^* ||. \tag{2.10.3}$$

Since $\mathfrak{X}^*$ is a (B)-space, it also has an adjoint space $\mathfrak{X}^{**}$. This is made up of the linear bounded functionals defined on $\mathfrak{X}^*$. Formula (2.10.2) shows that for fixed x we have $B(x, x^*) \in \mathfrak{X}^{**}$, that is, there exists an $x^{**} \in \mathfrak{X}^{**}$ with

$$x^{**}(x^*) = x^*(x) \tag{2.10.4}$$

for all x^*. By formula (2.10.3) we have $|| x^{**} || \leqq || x ||$. Here, however, the equality must hold; this is shown by Theorem 2.9.3 according to which there is a functional $x^* \in \mathfrak{X}^*$ of norm unity whose value at a given point x equals $|| x ||$.

If $x = \theta$, we have $B(x, x^*) = 0$ for all x^*. Conversely, by Theorem 2.9.3, if $B(x, x^*) = 0$ for all x^* and x is fixed, then $x = \theta$. This has the following important consequence:

THEOREM 2.10.1. *If $B(x, x^*) = B(y, x^*)$ for every x^*, then $x = y$.*

In other words, *if x and y are distinct elements of $\mathfrak{X}$, then there is at least one functional x^* such that $x^*(x) \neq x^*(y)$.*

The correspondence $x \to B(x, x^*)$ establishes a mapping of $\mathfrak{X}$ onto a subset $\mathfrak{X}_2$ of $\mathfrak{X}^{**}$. *The correspondence is an isomorphic and isometric homeomorphism*: it is one-to-one by Theorem 2.10.1, if $x \leftrightarrow x^{**}$, $y \leftrightarrow y^{**}$, then $\alpha x + \beta y \leftrightarrow \alpha x^{**} + \beta y^{**}$, and $|| x || = || x^{**} ||$. It follows that *the space $\mathfrak{X}$ may be embedded in the space $\mathfrak{X}^{**}$ without change of algebraic or metric relations.* This is the sense to be attached to

THEOREM 2.10.2. $\mathfrak{X} \subset \mathfrak{X}^{**}$.

The case in which $\mathfrak{X} = \mathfrak{X}^{**}$ is of particular interest. Such spaces were originally called *regular* by H. Hahn in 1927; following E. R. Lorch (1939) we shall use the more suggestive term *reflexive*. We list some properties of reflexive spaces in the next theorem, referring to B. J. Pettis [**2, 3**] for the proofs.

THEOREM 2.10.3. (1) *A closed linear subspace of a reflexive space is reflexive.* (2) *If $\mathfrak{X}$ is reflexive, so is $\mathfrak{X}^*$ and vice versa.* (3) *A uniformly convex space is reflexive.*

We note finally

THEOREM 2.10.4. *If $\mathfrak{X}^*$ is separable, so is $\mathfrak{X}$.*

PROOF. It is easily seen that if X is a countable set in a (B)-space $\mathfrak{X}$, then the least closed linear subspace containing X is separable. The assumption that $\mathfrak{X}^*$ is separable implies the existence of a countable set $\{x_n^*\}$ dense on the

surface of the unit sphere in $\mathfrak{X}^*$. Since $\| x_n^* \| = 1$, it is possible to choose for each n an element x_n of $\mathfrak{X}$ such that $\| x_n \| \leqq 1$ and $| x_n^*(x_n) | \geqq \frac{1}{2}$. If the set $\{x_n\}$ is not fundamental, Theorem 2.9.4 asserts the existence of a functional x^* with $\| x^* \| = 1, x^*(x_n) = 0$ for all n. We have then

$$\| x^* - x_n^* \| \geqq | (x^* - x_n^*)(x_n) | = | x_n^*(x_n) | \geqq \tfrac{1}{2},$$

so that $\{x_n^*\}$ would not be dense on the unit sphere of $\mathfrak{X}^*$ against the assumption. It follows that $\{x_n\}$ is fundamental and hence that $\mathfrak{X}$ is separable. The converse of this theorem is false: the space L is separable, but its adjoint space L_∞ is not. It should be noted that L is not reflexive.

2.11. The weak topology. The normed topology of a (B)-space is often called the *strong topology*. When this terminology is used, a Cauchy sequence $\{x_n\}$ is said to be *strongly convergent* and its limit is called the *strong limit* of x_n. In the applications it is frequently necessary to use also another topology, called the *weak topology*, in terms of which the space is a *Hausdorff linear space*.

The neighborhoods are defined as follows. Let $x_0 \in \mathfrak{X}$, $\epsilon > 0$, n arbitrary positive integer, and $x_1^*, x_2^*, \cdots, x_n^*$ any n elements of $\mathfrak{X}^*$. The set of all elements x such that

$$| x_k^*(x - x_0) | < \epsilon, \qquad k = 1, 2, \cdots, n, \tag{2.11.1}$$

constitutes the neighborhood $N(x_0\,; x_1^*, \cdots, x_n^*\,; \epsilon)$ of x_0. The set of all such neighborhoods is the system $\{N_\alpha\}$ which defines the topology. It is obvious that Hausdorff's axiom H_1 is satisfied. If N_1 and N_2 are two neighborhoods

$$N_1 = N(x_0\,; x_1^*, \cdots, x_m^*\,; \epsilon_1),\ N_2 = N(x_0\,; x_{m+1}^*, \cdots, x_{m+n}^*\,; \epsilon_2)$$

then

$$N_3 = N(x_0\,; x_1^*, \cdots, x_m^*, x_{m+1}^*, \cdots, x_{m+n}^*\,; \epsilon_3),$$

with $\epsilon_3 < \min(\epsilon_1, \epsilon_2)$, has the property $N_3 \subset N_1 \cap N_2$ so that axiom H_2 is satisfied. If $y_0 \in N(x_0)$, we take

$$N(y_0) = N(y_0\,; x_1^*, \cdots, x_n^*\,; \delta)$$

with the same functionals as in $N(x_0)$ and $\delta < \epsilon - \max_k | x_k^*(x_0 - y_0) |$. This condition ensures that $N(y_0) \subset N(x_0)$ so that axiom H_3 holds. Finally, in order to verify axiom H_4, we take

$$N(x_0) = N(x_0\,; x^*; \epsilon), N(y_0) = N(y_0\,; x^*; \epsilon)$$

where x^* is arbitrary except for the condition $x^*(y_0 - x_0) = 1$. This gives $x^*(x - x_0) - x^*(x - y_0) = 1$. Hence if $\epsilon < \frac{1}{2}$, we have $N(x_0) \cap N(y_0) = \emptyset$. It is also a simple matter to verify that axioms AH and SH of Definitions 1.8.1 and 1.10.1 are satisfied. It follows that in terms of this weak topology the (B)-space $\mathfrak{X}$ is a Hausdorff linear space. Thus $x + y$ and αx are continuous functions of (x, y) and (α, x) respectively, just as in the strong topology.

There are some surprising differences between the two topologies, however. Thus $\| x \|$ is not a continuous function of x in the weak topology. This follows from Theorem 2.9.7 according to which we can find an element y of arbitrarily large norm such that $x_1^*(y) = 0, \cdots, x_n^*(y) = 0$. It follows that every neighborhood $N(x_0; x_1^*, \cdots, x_n^*; \epsilon)$ contains elements of the form $x_0 + y$ where $\| y \|$ is arbitrarily large. This excludes continuity of the norm of x in this topology whenever $\mathfrak{X}$ is of infinite dimension.

The weak topology of Hilbert space was introduced by J. von Neumann in 1929, but the corresponding notion of weak convergence is much older. For the case of Lebesgue spaces it goes back to F. Riesz (1910).

Definition 2.11.1. *A sequence $\{x_n\}$ in a (B)-space $\mathfrak{X}$ is said to be weakly convergent if $\lim_{n\to\infty} x^*(x_n)$ exists for every $x^* \in \mathfrak{X}^*$; it is said to be weakly convergent to the element x_0 if $\lim_{n\to\infty} x^*(x_n) = x^*(x_0)$ for all x^*.*

Definition 2.11.2. *A (B)-space is said to be weakly complete in case every weakly convergent sequence in $\mathfrak{X}$ is weakly convergent to an element of $\mathfrak{X}$.*

Definition 2.11.3. *A subset X of a (B)-space is said to be sequentially weakly compact if every sequence in X contains a subsequence which converges weakly to a point in $\mathfrak{X}$.*

Strong convergence implies weak convergence, but not conversely.

Theorem 2.11.1. *If a sequence $\{x_n\}$ converges weakly, then there is an M such that $\| x_n \| \leqq M$ for all n. More generally, a sequentially weakly compact subset of $\mathfrak{X}$ is necessarily bounded.*

This is a special case of Theorem 2.12.3 below.

J. von Neumann [3] has called attention to the fact that weak closure cannot be defined in terms of weak convergence. In Hilbert space a sequence may have a single limit point in the weak topology, but nevertheless there is no subsequence which converges weakly to this point. This implies that the neighborhoods in the weak topology do not satisfy the first countability axiom, much less the second. A Hilbert space is not sequentially weakly compact, but bounded subsets have this property and this extends to reflexive spaces as shown by B. J. Pettis [2]:

Theorem 2.11.2. *A set in a reflexive space is sequentially weakly compact if and only if it is bounded in the strong topology.*

Theorem 2.11.3. *A reflexive space is weakly complete.*

References. Banach [2], Bohnenblust and Sobczyk [1], Day [1], Hahn [1, 2], Helly [1], Kolmogoroff [3], LaSalle [1], Lorch [1], v. Neumann [3], Pettis [2, 3], F. Riesz [1, 2], and Wehausen [1].

4. Linear Bounded Transformations

2.12. General principles. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be (B)-spaces with the same scalar field. Definitions 2.6.2 and 2.7.1 give

Definition 2.12.1. *A transformation $y = T(x)$ with domain $\mathfrak{X}$ and range in $\mathfrak{Y}$ is said to be linear and bounded if* (i) $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ *for all* $\alpha, \beta \in \Phi$ *and* $x_1, x_2 \in \mathfrak{X}$, (ii) $\| T(x) \| \leqq M \| x \|$ *for all* x. *The norm of T is* $\| T \| = \sup \| T(x) \|$ *for* $\| x \| \leqq 1$.

In this section we collect some basic theorems, which are used constantly, and start with the *Banach-Steinhaus theorem.*

Theorem 2.12.1. *Let $\{T_n\}$ be a sequence of linear bounded transformations on $\mathfrak{X}$ to $\mathfrak{Y}$ such that* (i) $\| T_n \| \leqq M$ *for all n, and* (ii) $\lim_{n\to\infty} T_n(x)$ *exists for every x in a set X which is dense in a sphere S. Then* $\lim_{n\to\infty} T_n(x)$ *exists for all x and the limit is a linear transformation with bound $\leqq M$.*

Proof. Cf. S. Banach [**2**, pp. 79–80]. The first step is to prove convergence in S. Let $x_0 \in S$ and $x_n \to x_0$, $x_n \in X$. For arbitrary positive integers n, p, q we have

$$\| T_p(x_0) - T_q(x_0) \| \leqq \| T_p(x_0) - T_p(x_n) \| + \| T_p(x_n) - T_q(x_n) \| + \| T_q(x_n) - T_q(x_0) \|,$$

whence

$$\limsup_{p,q\to\infty} \| T_p(x_0) - T_q(x_0) \| \leqq 2M \| x_0 - x_n \|.$$

Since $x_n \to x_0$, the convergence of $\{T_n(x_0)\}$ is proved. If y_0 is the center of S, r its radius, and x an arbitrary point in $\mathfrak{X}$, then $y_0 + \alpha x \in S$ for $| \alpha | < r/ \| x \|$, and consequently the sequence $\{T_n(y_0 + \alpha x)\}$ converges. But

$$T_n(y_0 + \alpha x) = T_n(y_0) + \alpha T_n(x). \tag{2.12.1}$$

Hence $\{T_n(x)\}$ also converges, that is, $T(x) = \lim_{n\to\infty} T_n(x)$ exists for all x. Since $T_n(x)$ is linear, $T(x)$ will have the same property and $\| T_n \| \leqq M$ implies $\| T \| \leqq M$.

For the following lemma we refer to S. Banach [**2,** pp. 13 and 19].

Lemma 2.12.1. *If $\{F_n(x)\}$ is a sequence of functionals defined on a (B)-space $\mathfrak{X}$ and if* $\limsup_{n\to\infty} | F_n(x) | < \infty$ *for all x in a set X which is of the second category in $\mathfrak{X}$, then there exists a sphere S in $\mathfrak{X}$ and an integer N such that $| F_n(x) | \leqq N$ for all x in S and all n.*

Next we take *the principle of uniform boundedness.*

THEOREM 2.12.2. *If* $\{T_\sigma(x)\}$ *is a set of linear bounded transformations on* $\mathfrak{X}$ *to* $\mathfrak{Y}$ *and if* $\sup_\sigma \| T_\sigma(x) \| < \infty$ *for all* x, *then the set* $\{ \| T_\sigma \| \}$ *is bounded.*

PROOF. We consider first the case in which the given set is countable, $T_\sigma = T_n$, $n = 1, 2, 3, \cdots$. By virtue of Lemma 2.12.1 there is a sphere S and an integer N such that $\| T_n(x) \| \leqq N$ for all x in S and all n. Let S be $\| x - y_0 \| < r$ and use formula (2.12.1). It shows that $| \alpha | \, \| T_n(x) \| \leqq 2N$ for $| \alpha | < r / \| x \|$. It follows that $\| T_n(x) \| \leqq (2N/r) \| x \|$ for all x and so $\| T_n \| \leqq 2N/r$ for all n. Supposing the theorem to be false for some non-countable set $\{T_\sigma\}$, we can find a countable subset $\{T_n\} \subset \{T_\sigma\}$ such that $\| T_n \| \to \infty$. This, however, leads to a contradiction with the countable case, since obviously $\sup_n \| T_n(x) \| \leqq \sup_\sigma \| T_\sigma(x) \| < \infty$ implies that $\sup_n \| T_n \|$ is finite.

Among the many consequences of this theorem, we note the following of which Theorem 2.11.1 is a special case.

THEOREM 2.12.3. *If a set of points* $\{x_\alpha\}$ *of* $\mathfrak{X}$ *is such that* $\sup_\alpha | x^*(x_\alpha) | < \infty$ *for all functionals* x^* *of* $\mathfrak{X}^*$, *then the set is bounded.*

PROOF. We use the second adjoint space. By Theorem 2.10.2 there is to every $x_\alpha \in \mathfrak{X}$ an element $x_\alpha^{**} \in \mathfrak{X}^{**}$ with (i) $x_\alpha^{**}(x^*) = x^*(x_\alpha)$ for all x^* and (ii) $\| x_\alpha \| = \| x_\alpha^{**} \|$. The set $\{x_\alpha^{**}(x^*)\}$ of linear bounded transformations on $\mathfrak{X}^*$ to the complex plane satisfies the conditions of Theorem 2.12.2. Hence the set $\{ \| x_\alpha^{**} \| \} = \{ \| x_\alpha \| \}$ is bounded.

2.13. Inverse and adjoint transformations. As above $\mathfrak{X}$ and $\mathfrak{Y}$ are (B)-spaces with the same scalar field Φ; $y = T(x)$ is a linear transformation, not necessarily bounded, with domain $\mathfrak{D} \subset \mathfrak{X}$ and range $\mathfrak{R} \subset \mathfrak{Y}$. The basic theorems on the inverse and the adjoint transformations will be presented in this section; practically all the results are due to S. Banach and so are the proofs.

We recall that $y = T(x)$ has an inverse $x = T^{-1}(y)$ if and only if the correspondence between $\mathfrak{D}$ and $\mathfrak{R}$ is one-to-one. In this case

$$y = T[T^{-1}(y)], \qquad y \in \mathfrak{R}; \; x = T^{-1}[T(x)], \qquad x \in \mathfrak{D}.$$

The condition for the existence of the inverse is that $x_1 \neq x_2$ should imply $T(x_1) \neq T(x_2)$ and, owing to the linearity, this is equivalent to

$$T(x) = \theta \text{ if and only if } x = \theta. \tag{2.13.1}$$

A simple argument shows that *the inverse of a linear transformation is also linear.*

THEOREM 2.13.1. *A necessary and sufficient condition that* $y = T(x)$ *have a bounded inverse is the existence of an* $m > 0$ *such that* $\| T(x) \| \geqq m \| x \|$ *for all* x *in* $\mathfrak{D}$. *The largest admissible value of* m *is the reciprocal of the norm of* T^{-1}.

PROOF. If $T^{-1}(y)$ exists as a bounded transformation, then $\| x \| = \| T^{-1}(y) \| \leqq M \| y \|$, $y \in \mathfrak{R}$, and $\| T(x) \| \geqq (1/M) \| x \|$. Conversely, if $\| T(x) \| \geqq m \| x \|$, then (2.13.1) holds, $T^{-1}(y)$ exists, and $\| T^{-1}(y) \| \leqq (1/m) \| y \|$.

THEOREM 2.13.2. *If $y = T(x)$ and its inverse are bounded linear transformations, and if the domain of $T(x)$ is closed, then the range is also closed.*

PROOF. Given $\lim y_n = y_0$, $y_n \in \mathfrak{R}$, and $y_n = T(x_n)$, then $\| x_m - x_n \| \leqq \| T^{-1} \| \| y_m - y_n \|$ whence it follows that $\lim x_n = x_0$ exists. Since $T(x)$ is defined and continuous for $x = x_0$, we have $T(x_0) = \lim T(x_n) = y_0$, that is, $y_0 \in \mathfrak{R}$ and $\mathfrak{R}$ is closed.

DEFINITION 2.13.1. *Let $y = T(x)$ be a bounded linear transformation and let $\mathfrak{X}^*$ and $\mathfrak{Y}^*$ be the adjoint spaces of $\mathfrak{X}$ and $\mathfrak{Y}$ respectively. Set $x^*(x) = y^*[T(x)]$ for $y^* \in \mathfrak{Y}^*$ and $x^* = T^*(y^*)$. The transformation T^* on $\mathfrak{Y}^*$ to $\mathfrak{X}^*$ is called the adjoint of T.*

It is obvious that $y^*[T(x)] \in \mathfrak{X}^*$ for every y^*.

THEOREM 2.13.3. *T^* is a linear bounded transformation and $\| T^* \| = \| T \|$.*

PROOF. Since $| x^*(x) | \leqq \| y^* \| \| T \| \| x \|$, we have $\| T^* \| \leqq \| T \|$. But by the definition of the norm, we may find an x_0 with $\| x_0 \| = 1$ and $\| T(x_0) \| > \| T \| - \epsilon$ for any preassigned $\epsilon > 0$. If $T(x_0) = y_0$, we take the functional $y^* \in \mathfrak{Y}^*$ such that $y^*(y_0) = \| y_0 \|$, $\| y^* \| = 1$, which exists by Theorem 2.9.3. Then $x^*(x_0) = y^*(y_0) = \| y_0 \| \geqq \| T \| - \epsilon$, so that $\| T^* \| > \| T \| - \epsilon$. Hence $\| T^* \| = \| T \|$. The linearity is obvious.

DEFINITION 2.13.2. *If $T_1(x)$ and $T_2(x)$ are linear transformations on $\mathfrak{X}$ to $\mathfrak{Y}$ with domains $\mathfrak{D}_1$ and $\mathfrak{D}_2$ such that $\mathfrak{D}_1 \subset \mathfrak{D}_2$ and $T_1(x) = T_2(x)$ in $\mathfrak{D}_1$, then $T_2(x)$ is called an extension of $T_1(x)$.*

THEOREM 2.13.4. *If $y = T(x)$ is a linear bounded transformation on $\mathfrak{X}$ to $\mathfrak{Y}$, then its second adjoint T^{**} is an extension of T defined on $\mathfrak{X}^{**}$ to $\mathfrak{Y}^{**}$ and $\| T^{**} \| = \| T \|$. In particular, $T^{**} = T$ if $\mathfrak{X}$ is a reflexive space.*

PROOF. The second adjoint is defined as $T^{**} = (T^*)^*$. It follows from the preceding theorem that T^{**} is a linear bounded transformation on $\mathfrak{X}^{**}$ to $\mathfrak{Y}^{**}$ and that $\| T^{**} \| = \| T^* \| = \| T \|$. By Theorem 2.10.1, $\mathfrak{X} \subset \mathfrak{X}^{**}$. Now take an $x^{**} = x \in \mathfrak{X}$ and let $y^{**} = T^{**}(x)$. We shall show that $y^{**} = T(x)$. This follows from a repeated use of the definition of the adjoint. Taking an arbitrary $y^* \in \mathfrak{Y}^*$, we have

$$y^{**}(y^*) = [T^{**}(x)](y^*) = x(x^*) = x[T^*(y^*)] = [T(x)](y^*).$$

Since this holds for every y^*, we have $y^{**} = T(x)$. This shows that T^{**} coincides with T in $\mathfrak{X}$, so that T^{**} is actually an extension of T. If $\mathfrak{X}$ is reflexive, $\mathfrak{X} = \mathfrak{X}^{**}$, and consequently $T^{**} = T$.

We have seen that the adjoint preserves boundedness. J. Schauder [1] has proved the sharper result: *if T is a linear compact transformation* (see Definition 2.3.3) *so is T^*.*

We now turn to the use of the adjoint transformation for the study of the inverse.

THEOREM 2.13.5. *A linear bounded transformation has a linear bounded inverse if and only if the range of its adjoint is the whole space* $\mathfrak{X}^*$.

PROOF. If every element $x^* \in \mathfrak{X}^*$ is of the form $T^*(y^*)$ for a suitable $y^* \in \mathfrak{Y}^*$ with $x^*(x) = y^*[T(x)]$, then $T(x_1) \neq T(x_2)$ when $x_1 \neq x_2$, since otherwise $x^*(x_1) = x^*(x_2)$ for every x^*, which is clearly impossible. Hence $T^{-1}(y)$ exists. If T^{-1} is not continuous, there would exist sequences $\{y_n\} \subset \mathfrak{Y}$ and $\{x_n\} \subset \mathfrak{X}$ such that (i) $y_n = T(x_n)$, (ii) $\|y_n\| \to 0$, and (iii) $\|x_n\| \to \infty$. For an arbitrary $y^* \in \mathfrak{Y}^*$ and the corresponding x^* we have $x^*(x_n) = y^*(y_n) \to 0$ when $n \to \infty$. Since this must hold for every $x^* \in \mathfrak{X}^*$, it follows from Theorem 2.12.3 that $\|x_n\| \leqq M$, against our assumption. Hence $T^{-1}(y)$ is continuous.

Conversely, if $T^{-1}(y)$ exists and is continuous, then for every $x^* \in \mathfrak{X}^*$ the formula $x^*[T^{-1}(y)] \equiv y^*(y), y \in \mathfrak{R}$, defines an element of $\mathfrak{Y}^*$. Further $y^*[T(x)] = x^*\{T^{-1}[T(x)]\} = x^*(x)$ so that $x^* = T^*(y^*)$. Since x^* is arbitrary, it follows that the range of T^* is $\mathfrak{X}^*$.

THEOREM 2.13.6. *The closure of the range of the linear bounded transformation* $y = T(x)$ *consists of all points* y *such that* $T^*(y^*) = \theta$ *implies* $y^*(y) = 0$.

PROOF. Suppose that $y_0 \in \mathfrak{Y}$ is such that $y^*(y_0) = 0$ for every y^* satisfying $T^*(y^*) = \theta$. If y_0 is not in $\overline{\mathfrak{R}}$, then Theorem 2.9.4 shows the existence of a functional y_0^* such that $y_0^*(y_0) = 1, y_0^*(\overline{\mathfrak{R}}) = 0$. If $x_0^* = T^*(y_0^*)$, we have $x_0^*(x) = y_0^*[T(x)] = 0$ for all $x \in \mathfrak{X}$, whence it follows that $x_0^* = \theta$ and $T^*(y_0^*) = \theta$. By assumption this requires $y_0^*(y_0) = 0$ in contradiction to the definition of y_0^*. Hence $y_0 \in \overline{\mathfrak{R}}$.

Let $\mathfrak{R}_0$ denote the set of points y satisfying the conditions of the problem, that is, such that $y^*(y) = 0$ for every y^* satisfying $T^*(y^*) = \theta$. Since $\mathfrak{R}_0$ is the intersection of closed sets, $\mathfrak{R}_0$ is closed. Now suppose that $y_1 \in \mathfrak{R}$ and that $T^*(y^*) = \theta$ so that the associated functional $x^* = \theta$. Then $y^*(y_1) = x^*(x_1) = 0$ where $T(x_1) = y_1$. It follows that $\mathfrak{R} \subset \mathfrak{R}_0$ and hence $\mathfrak{R}_0 = \overline{\mathfrak{R}}$.

We shall not press the investigation of the properties of the adjoint any further. The remaining theorems in this section are concerned with the notion of closed transformations and the properties of the inverse. Theorem 2.13.7 (in a slightly less general form with "bounded" instead of "closed") was proposed by Nelson Dunford in lieu of a previous argument of the author's.

DEFINITION 2.13.3. *A linear transformation is said to be closed in case* $\lim_{n\to\infty} x_n = x_0$ *and* $\lim_{n\to\infty} y_n = y_0$, *with* $x_n \in \mathfrak{D}, y_n = T(x_n)$, *always implies that* $x_0 \in \mathfrak{D}$ *and* $T(x_0) = y_0$.

THEOREM 2.13.7. *If* $y = T(x)$ *is a closed linear transformation whose range* $\mathfrak{R}$ *is of the second category in* $\mathfrak{Y}$, *then*

(1) $\mathfrak{R} = \mathfrak{Y}$;

(2) *there is a constant* $m > 0$ *such that to every* $y \in \mathfrak{Y}$ *there is an* $x \in \mathfrak{D}$ *with* $y = T(x)$ *and* $\|x\| \leqq m\|y\|$;

(3) *if* T^{-1} *exists then it is bounded.*

Proof. Since $\mathfrak{R}$ is of the second category in the (B)-space $\mathfrak{Y}$, there exists an open sphere $K_0 \subset \mathfrak{Y}$ at each point of which $\mathfrak{R}$ is of the second category for Int [D $(\mathfrak{R})$] $\neq \emptyset$, see section 1.2. Let $r > 0$ be fixed but arbitrary, let S be the sphere $\| x \| < r$, and set $\mathfrak{D}_1 = \mathfrak{D} \cap S$. Then

$$\mathfrak{R} \cap K_0 = \bigcup_1^\infty [T(\tfrac{1}{4}n\mathfrak{D}_1) \cap K_0]$$

and there is an integer n for which $T(\frac{1}{4}n\mathfrak{D}_1) \cap K_0$ is of the second category. Hence there is an open sphere $K \subset K_0$ at each point of which $T(\frac{1}{4}n\mathfrak{D}_1)$ is of the second category. Since $D(X) \subset \bar{X}$, we have $\overline{T(\frac{1}{4}n\mathfrak{D}_1)} \supset K$. Hence

$$\overline{T(\tfrac{1}{2}\mathfrak{D}_1)} \supset \overline{T(\tfrac{1}{4}\mathfrak{D}_1) - T(\tfrac{1}{4}\mathfrak{D}_1)} \supset \overline{T(\tfrac{1}{4}\mathfrak{D}_1)} - \overline{T(\tfrac{1}{4}\mathfrak{D}_1)} \supset \frac{1}{n} K - \frac{1}{n} K \supset S_1$$

where S_1 is an open sphere, $\| y \| < r_1$ say. It follows that

$$\overline{T(2^{-1-k}\mathfrak{D}_1)} \supset 2^{-k} S_1 , \qquad k = 0, 1, 2, \cdots . \tag{2.13.2}$$

Let y be an arbitrary point of S_1. Using (2.13.2) with $k = 0$, we can find a point $x_1 \in 2^{-1}\mathfrak{D}_1$ such that

$$\| y - T(x_1) \| < 2^{-1} r_1, \quad \| x_1 \| < 2^{-1} r.$$

Since $y - T(x_1) \in 2^{-1}S_1$, we can use (2.13.2) with $k = 1$ to pick a point $x_2 \in 2^{-2}\mathfrak{D}_1$ such that

$$\| y - T(x_1) - T(x_2) \| < 2^{-2} r, \| x_2 \| < 2^{-2} r.$$

By using (2.13.2) repeatedly in this manner, we obtain a sequence $\{x_n\} \in \mathfrak{D}_1$ with

$$\| y - T(x_1) - \cdots - T(x_n) \| = \| y - T(\sum_1^n x_k) \| < 2^{-n} r_1 , \| x_n \| < 2^{-n} r.$$

Upon setting $s_n = \sum_1^n x_k$, we see that (i) $s_n \in \mathfrak{D}_1$, (ii) $\lim_{n\to\infty} s_n = x$ exists and is in S, (iii) $\lim_{n\to\infty} T(s_n) = y$. Since T is closed it follows that $x \in \mathfrak{D}_1$ and $y = T(x)$.

Thus we have shown that the image $T(\mathfrak{D}_1)$ contains a sphere S_1 with center at θ. Statement (1) is an immediate consequence of this fact. Statement (2) follows by taking $m = 2r/r_1$ for if $y \in \mathfrak{Y}$ we have $r_1 y/(2 \| y \|) \in S_1$ and hence there is an $x_0 \in \mathfrak{D}_1$ such that $\| x_0 \| < r$, $r_1 y/(2 \| y \|) = T(x_0)$ and thus, if $x = (2 \| y \| / r_1) x_0$ we have $y = T(x)$ and $\| x \| \leqq (2r/r_1) \| y \|$. Statement (3) follows from (2) and Theorem 2.13.1. This completes the proof.

Corollary. *If $y = T(x)$ is a linear bounded transformation mapping $\mathfrak{X}$ in a one-to-one manner onto $\mathfrak{Y}$, then the inverse is also bounded.*

For a bounded transformation is closed.

If in addition to the assumptions of Theorem 2.13.7 we know that $\mathfrak{D} = \mathfrak{X}$, then we can conclude that T is bounded. This follows from Theorem 2.13.9 below but can also be proved

as follows. If T has an inverse, we have merely to apply the theorem to T^{-1} in order to see that $(T^{-1})^{-1} = T$ is bounded. We can reduce the general case to this particular one by a device which may be useful also for other purposes. We say that x_1 *and* x_2 *belong to the same equivalence class, modulo* T, *if* $T(x_1) = T(x_2)$. Since T is defined for all x and is closed, every x belongs to a unique equivalence class X and these classes are closed point sets in $\mathfrak{X}$. The equivalence classes form a linear system in the sense of Definition 1.9.1 and this system can be made into a (B)-space $\mathfrak{X}_0$ by defining $\| X \| = \inf \| x \|$, $x \in X$. Cf. the proof of Theorem 22.11.4 (the Gelfand-Neumark metric); the idea of metrizing equivalence classes in metric spaces is due to S. Banach [2, p. 232] and F. Hausdorff. To the transformation T on $\mathfrak{X}$ to $\mathfrak{Y}$ corresponds a transformation T_0 on $\mathfrak{X}_0$ to $\mathfrak{Y}$ defined by $T_0(X) = T(x)$ for $x \in X$. Since T is closed, so is T_0. Further T_0^{-1} exists since $T_0(X) = \theta$ if and only if $X = \theta$. Theorem 2.13.7 then applies to T_0^{-1} and shows that T_0 is bounded. If x and z are elements of X, we have

$$\| T(x) \| = \| T_0(X) \| \leqq M \| X \| = M \inf \| z \| \leqq M \| x \|,$$

so that T is also bounded.

THEOREM 2.13.8. *If a linear system can be made into a (B)-space by two different choices of a norm,* $\| x \|_1$ *and* $\| x \|_2$, *in such a manner that* $\lim_{n\to\infty} \| x_n \|_2 = 0$ *always implies* $\lim_{n\to\infty} \| x_n \|_1 = 0$, *then the corresponding notions of convergence are equivalent and for* $x \neq \theta$ *we have* $0 < m \leqq \| x \|_2 / \| x \|_1 \leqq M < \infty$, *where* m *and* M *are independent of* x.

PROOF. This is an immediate consequence of the corollary of the preceding theorem if one takes for $\mathfrak{X}$ and $\mathfrak{Y}$ the (B)-spaces corresponding to the two norms and sets $T(x) = x$.

We come finally to the *closed graph theorem.*

THEOREM 2.13.9. *A closed linear transformation which is defined everywhere in* $\mathfrak{X}$ *is bounded.*

PROOF. We introduce a second norm in $\mathfrak{X}$ by putting $\| x \|_2 = \| x \| + \| T(x) \|$, $\| x \|_1 = \| x \|$. This definition leads to a new metric in $\mathfrak{X}$ and we shall show that the space remains complete. Suppose that $\lim_{m,n} \| x_m - x_n \|_2 = 0$. This implies $\lim_{m,n} \| x_m - x_n \|_1 = 0$ and $\lim_{m,n} \| T(x_m) - T(x_n) \| = 0$. Since both $\mathfrak{X}$ and $\mathfrak{Y}$ are complete, there are points $x_0 \in \mathfrak{X}$, $y_0 \in \mathfrak{Y}$ such that $\| x_n - x_0 \|_1 \to 0$, $\| T(x_n) - y_0 \| \to 0$. $T(x)$ being closed, we have $T(x_0) = y_0$. It follows that $\| x_n - x_0 \|_2 \to 0$, so that $\mathfrak{X}$ is a (B)-space also in the second metric. Since $\| x \|_1 \leqq \| x \|_2$, the assumptions of the preceding theorem are satisfied and the resulting inequality $\| x \|_2 \leqq M \| x \|_1$ asserts that $T(x)$ is bounded.

The graph of the transformation $y = T(x)$ is the point set $[x, T(x)]$, $x \in \mathfrak{D}$, in the product space $\mathfrak{X} \times \mathfrak{Y}$. We norm the latter by setting $\| (x, y) \| = \| x \| + \| y \|$. Definition 2.13.3 then gives

THEOREM 2.13.10. *A transformation is closed if and only if its graph is closed.*

If T^{-1} exists, we see that the graph may also be written in the form $[T^{-1}(y), y]$, $y \in \mathfrak{R}$. This observation leads to

THEOREM 2.13.11. *If a closed linear transformation has an inverse, the inverse is also closed.*

2.14. Resolvents and spectra. In the present section we consider a linear transformation, not necessarily bounded, whose domain $\mathfrak{D}$ and range $\mathfrak{R}$ are located in the same complex (B)-space $\mathfrak{X}$. We shall make a preliminary study of the spectral properties of T. The resolvent will be studied in more detail and from a different angle in §5.3.

The transformation $T_\lambda = \lambda I - T$, where λ is an arbitrary complex number and I is the identity transformation, is also defined in $\mathfrak{D}$. Its range will be denoted by $\mathfrak{R}_\lambda$. The distribution of the values of λ for which T_λ has an inverse and the properties of the inverse when it exists are basic in the description of T.

DEFINITION 2.14.1. *The values of λ for which T_λ has a bounded inverse $R_\lambda = R(\lambda; T)$ with domain dense in $\mathfrak{X}$ form the resolvent set $\rho(T)$ of T. $R(\lambda; T)$ is called the resolvent of T. The values of λ for which T_λ has an unbounded inverse with domain dense in $\mathfrak{X}$ form the continuous spectrum $C\sigma(T)$. The values of λ for which T_λ has an inverse whose domain is not dense in $\mathfrak{X}$ form the residual spectrum $R\sigma(T)$. The values of λ for which no inverse exists form the point spectrum $P\sigma(T)$. The union of $C\sigma(T)$, $P\sigma(T)$, and $R\sigma(T)$ is the spectrum $\sigma(T)$ of T.*

The definition gives

THEOREM 2.14.1. *The four sets $\rho(T)$, $C\sigma(T)$, $P\sigma(T)$, and $R\sigma(T)$ are mutually exclusive and their union is the complex plane.*

It should be noted that $\mathfrak{R}_\lambda$, the range of T_λ, is dense in $\mathfrak{X}$ if $\lambda \in \rho(T) \cup C\sigma(T)$, non-dense if $\lambda \in R\sigma(T)$, and is not subjected to any restriction if $\lambda \in P\sigma(T)$. Statement (2.13.1) gives

THEOREM 2.14.2. *A necessary and sufficient condition that $\lambda_0 \in P\sigma(T)$ is that the equation $T(x) = \lambda_0 x$ have a solution $x \neq \theta$.*

DEFINITION 2.14.2. *If $\lambda_0 \in P\sigma(T)$, then λ_0 is called a characteristic value of T and if $T(x) = \lambda_0 x$, $x \neq \theta$, then x is a characteristic element (vector) of T; it is normalized if $\| x \| = 1$. The least closed linear manifold $\mathfrak{M}(\lambda_0 ; T)$ containing the characteristic elements corresponding to λ_0 is called the characteristic manifold of T corresponding to the value λ_0.*

In connection with these notions the reader should consult Definitions 14.6.1 and 15.5.1 below.

THEOREM 2.14.3. *If $\lambda_0 \in C\sigma(T)$, there exists a sequence $\{x_n\} \in \mathfrak{D}$ such that* (i) $\| x_n \| = 1$, (ii) $\| T_{\lambda_0} (x_n) \| \to 0$.

This follows from Theorem 2.13.1. On the other hand, the existence of such a sequence indicates merely that $\lambda_0 \in \sigma(T)$ but not necessarily to $C\sigma(T)$.

The change of the spectrum under extensions of T is important. If T^0 is an extension with domain $\mathfrak{D}^0$, then $T_\lambda^0 = \lambda I - T^0$ is an extension of T_λ and if their ranges are $\mathfrak{R}_\lambda^0$ and $\mathfrak{R}_\lambda$ respectively, then $\mathfrak{D} \subset \mathfrak{D}_0$, $\mathfrak{R}_\lambda \subset \mathfrak{R}_\lambda^0$. If T_λ^0 has an

inverse, so does T_λ, and $(T_\lambda^0)^{-1}$ is an extension of $(T_\lambda)^{-1}$. On the other hand, the existence of $(T_\lambda)^{-1}$ does not imply that of $(T_\lambda^0)^{-1}$.

THEOREM 2.14.4. *If T^0 is an extension of T, then $P\sigma(T) \subseteqq P\sigma(T^0)$ and $\mathfrak{M}(\lambda_0 ; T) \subseteqq \mathfrak{M}(\lambda_0 ; T^0)$ for every $\lambda_0 \in P\sigma(T)$. Further, $R\sigma(T) \supseteqq R\sigma(T^0)$, $C\sigma(T) \subseteqq C\sigma(T^0) \cup P\sigma(T^0)$, and $\rho(T^0) \subseteqq \rho(T) \cup R\sigma(T)$.*

PROOF. The first statement is obvious. If $\lambda \in R\sigma(T^0)$, then $(T_\lambda^0)^{-1}$ exists and $\mathfrak{R}_\lambda^0$ is non-dense in $\mathfrak{X}$. Hence $(T_\lambda)^{-1}$ also exists and $\mathfrak{R}_\lambda$ is non-dense. This proves $R\sigma(T) \supseteqq R\sigma(T^0)$. On the other hand, a point in $R\sigma(T)$ may very well turn up in any one of the four sets associated with T^0. If $\lambda \in C\sigma(T)$, then $\mathfrak{R}_\lambda \subseteqq \mathfrak{R}_\lambda^0$ and both are dense in $\mathfrak{X}$; $(T_\lambda)^{-1}$ either does not exist or is unbounded since $(T_\lambda)^{-1}$ is unbounded. Finally, if $\lambda \in \rho(T^0)$, then $(T_\lambda^0)^{-1}$ exists as a bounded transformation and $\mathfrak{R}_\lambda^0$ is dense in $\mathfrak{X}$. It follows that $(T_\lambda)^{-1}$ also exists and is bounded, but its domain may be non-dense. This proves the last statement. Conversely, a point in $\rho(T)$ may turn up in any one of the sets associated with T^0 except in $R\sigma(T^0)$.

Thus an extension may enlarge the point spectrum. No characteristic value is ever lost, existing ones may get their characteristic manifolds enlarged but never contracted. New characteristic values may be generated from $\rho(T)$ or by transfer from the rest of $\sigma(T)$. The continuous spectrum may suffer losses to the point spectrum but may gain from the residual spectrum or from the resolvent set of T. Finally, the residual spectrum never gains and the resolvent set can gain only from the residual spectrum.

Anticipating a result of the discussion in sections 5.8 and 5.14 we state

THEOREM 2.14.5. *The resolvent set is open and the spectrum is never vacuous in the extended complex plane.*

It may happen, however, that the whole finite plane belongs to the resolvent set. In this case the point at infinity is assigned to the spectrum. This will be justified later.

References. Banach [1, 2], Lorch [2], Schauder [1], Stone [2], Taylor [2].

5. SPACES OF ENDOMORPHISMS

2.15. The Banach algebra of endomorphisms. The family of all linear bounded transformations on a (B)-space $\mathfrak{X}$ to itself is one of the most important instances of a Banach algebra in the sense of Definition 1.14.1. We introduce

DEFINITION 2.15.1. *A linear bounded transformation on a (B)-space $\mathfrak{X}$ to itself will be called an endomorphism of $\mathfrak{X}$.*

This use of the term is sufficiently close to modern usage in abstract algebra so that no confusion is likely to arise.

DEFINITION 2.15.2. *If T, T_1, and T_2 are endomorphisms of $\mathfrak{X}$, then $T_1 + T_2$, αT for $\alpha \in \Phi$, and T_1T_2 are defined for all x by*

$$(T_1 + T_2)(x) = T_1(x) + T_2(x), \quad (\alpha T)(x) = \alpha T(x), \quad (T_1T_2)(x) = T_1[T_2(x)].$$

THEOREM 2.15.1. *If T, T_1, and T_2 are endomorphisms of $\mathfrak{X}$, so are $T_1 + T_2$, αT, and $T_1 T_2$. Further*

$$\| T_1 + T_2 \| \leqq \| T_1 \| + \| T_2 \|, \; \| \alpha T \| = | \alpha | \, \| T \|,$$
$$\| T_1T_2 \| \leqq \| T_1 \| \, \| T_2 \|.$$

PROOF. This is obvious.

THEOREM 2.15.2. *The set of all endomorphisms of $\mathfrak{X}$ forms a Banach algebra $\mathfrak{E}(\mathfrak{X})$ which is normally non-commutative and has the identity transformation as unit element.*

PROOF. It is understood that the algebraic operations in $\mathfrak{E}(\mathfrak{X})$ are defined by Definition 2.15.2 and that the norm is defined by $\| T \| = \sup \| T(x) \|$, $\| x \| \leqq 1$, as usual. The only point that calls for comment is the completeness of the space. Let $\{T_n\}$ be a Cauchy sequence and set $\delta_{mn} = \| T_m - T_n \|$. Since $\| T_m(x) - T_n(x) \| \leqq \delta_{mn} \| x \|$ and $\mathfrak{X}$ is complete, the sequence $\{T_n(x)\}$ converges for all x to a limit $T_0(x)$ when $n \to \infty$. If $\delta_{mn} \leqq \epsilon$ for $m, n > N$, we have $\| T_n(x) - T_0(x) \| \leqq \epsilon \| x \|$ and $\| T_n - T_0 \| \leqq \epsilon$. It follows that when $n \to \infty$ we have $\| T_n - T_0 \| \to 0$ and $\mathfrak{E}(\mathfrak{X})$ is complete.

We call $\mathfrak{E}(\mathfrak{X})$ *the Banach algebra of endomorphisms of* $\mathfrak{X}$.

If T is a fixed element of $\mathfrak{E}(\mathfrak{X})$, a polynomial in T is of the form

$$\alpha_0 I + \sum_1^n \alpha_k T^k, \; T^k = TT^{k-1}, \qquad \alpha_k \in \Phi.$$

The set of all such polynomials in a fixed T is a commutative subalgebra of $\mathfrak{E}(\mathfrak{X})$ and so is its closure.

In the following the terms *operator* and *operation* will often be used as synonyms for transformation.

2.16. The strong and the weak topologies. The system of all endomorphisms of $\mathfrak{X}$ was made into a Banach algebra in the preceding section by the introduction of the *normed topology*, in this connection usually referred to as the *uniform topology*. Following J. von Neumann [3] we may also introduce *the strong and the weak topologies*. The resulting *topological algebras* will be denoted by $\mathfrak{E}_s(\mathfrak{X})$ and $\mathfrak{E}_w(\mathfrak{X})$ respectively. They are not Hausdorff algebras in the sense of Definition 1.13.1 because Postulate MH does not hold, though they are Haus-

dorff spaces and Postulates AH and SH are satisfied. These topologies correspond to well-known definitions of convergence.

DEFINITION 2.16.1. (1) *The sequence* $\{T_n\}$ *of endomorphisms of* $\mathfrak{X}$ *converges strongly if* $\lim_{m,n\to\infty} \| T_m(x) - T_n(x) \| = 0$ *and converges strongly to the endomorphism* T_0 *if* $\lim_{n\to\infty} \| T_n(x) - T_0(x) \| = 0$ *for all* x. (2) *The sequence converges weakly if* $\lim_{m,n\to\infty} | x^*[T_m(x)] - x^*[T_n(x)] | = 0$ *and it converges weakly to* T_0 *if* $\lim_{n\to\infty} | x^*[T_n(x)] - x^*[T_0(x)] | = 0$ *for all* $x \in \mathfrak{X}$ *and all* $x^* \in \mathfrak{X}^*$.

The corresponding *neighborhood topologies* are defined in a manner analogous to the procedure used in section 2.11 for introducing the weak neighborhoods in $\mathfrak{X}$. A *strong neighborhood* of T_0 is any set $N[T_0 ; x_1, \cdots, x_n ; \epsilon]$ which is made up of all operators T such that

$$\| T(x_k) - T_0(x_k) \| < \epsilon, \qquad k = 1, \cdots, n,$$

where $x_1, \cdots, x_n$ are n arbitrary elements of $\mathfrak{X}$. Similarly a *weak neighborhood* is any set $N[T_0 ; x_1, \cdots, x_n ; \overset{*}{x}_1, \cdots, \overset{*}{x}_n ; \epsilon]$ made up of the T's satisfying

$$| \overset{*}{x}_k[T(x_k)] - \overset{*}{x}_k[T_0(x_k)] | < \epsilon, \qquad k = 1, \cdots, n,$$

where $x_1, \cdots, x_n$ and $\overset{*}{x}_1, \cdots, \overset{*}{x}_n$ are arbitrary elements of $\mathfrak{X}$ and $\mathfrak{X}^*$ respectively. The set of all neighborhoods is obtained by varying T_0, ϵ, n, x_k, and $\overset{*}{x}_k$ over their domains of definition. We leave to the reader the verification of the fact that these two types of neighborhoods satisfy the Hausdorff axioms and that Postulates AH and SH of Definitions 1.8.1 and 1.10.1 hold as well. On the other hand, multiplication is continuous merely with respect to one factor at the time.

THEOREM 2.16.1. $\mathfrak{E}_s(\mathfrak{X})$ *is strongly complete in the sense that every strongly convergent sequence* $\{T_n\}$ *converges strongly to an element of the space.*

PROOF. Since $\mathfrak{X}$ is complete, $\| T_m(x) - T_n(x) \| \to 0$ for all x when $m, n \to \infty$ implies the existence of a $T_0(x)$ such that $\| T_n(x) - T_0(x) \| \to 0$ for all x. By Theorem 2.12.2 there is an M with $\| T_n \| \leqq M$ for all n and consequently $\| T_0 \| \leqq M$ so that $T_0 \in \mathfrak{E}_s(\mathfrak{X})$.

DEFINITION 2.16.2. $\mathfrak{E}_w(\mathfrak{X})$ *is said to be weakly complete if every weakly convergent sequence in* $\mathfrak{E}_w(\mathfrak{X})$ *converges weakly to an element of the space.*

Reference. v. Neumann [3].

CHAPTER III

VECTOR-VALUED FUNCTIONS

3.1. Orientation. This chapter is devoted to an exposition of *the theory of functions of real or complex variables having values in a (B)-space.* In §2.3 we considered functionals, that is, functions on vectors to numbers; here the dependence is reversed and we are concerned with functions on numbers to vectors. Related questions for functions having values in a Banach algebra are discussed in Chapter V if the operation of multiplication enters in the problem in an essential manner, otherwise they are included in the present chapter. The exposition still leans heavily on the work of other writers; material has been selected with a view of providing an account of the general ideas as well as the special results which will be needed in Part Two of this treatise. Neither the text nor the References aim at completeness. The paragraph headings: *Abstract Integration* and *Complex Function Theory*, indicate the main themes.

1. Abstract Integration

3.2. Functions and measure. Let E_k be the k-dimensional euclidean space, S a measurable set in E_k, and $x = x(\alpha)$ a mapping of S onto a set $\mathfrak{R}$ in a (B)-space $\mathfrak{X}$. Thus, to every point $\alpha = (\alpha_1, \cdots, \alpha_k)$ of S corresponds a unique vector $x(\alpha)$ in $\mathfrak{R}$ so that $x(\alpha)$ is a function on number k-tuples to vectors. Our first task in the study of such a function is to discuss the notion of *measurability*. Actually there are several such notions and a number of definitions are required at the outset.

Definition 3.2.1. *Let $x(\alpha)$ and $\{x_n(\alpha)\}$ be functions on S to $\mathfrak{X}$. The sequence $\{x_n(\alpha)\}$ converges to $x(\alpha)$ in S*

(1) *almost uniformly if to every $\epsilon > 0$ there is a set $S_\epsilon \subset S$ with $m(S_\epsilon) < \epsilon$ and to every $\delta > 0$ an integer $n(\delta)$ such that $\|x(\alpha) - x_n(\alpha)\| < \delta$ for $\alpha \in S - S_\epsilon$ and $n > n(\delta)$;*

(2) *almost everywhere if there exists a null set S_0 in S such that $\| x(\alpha) - x_n(\alpha) \| \to 0$ for $\alpha \in S - S_0$ when $n \to \infty$;*

(3) *in measure if for every $\epsilon > 0$ the* outer *measure of the subset of S where $\| x(\alpha) - x_n(\alpha) \| > \epsilon$ tends to zero when $n \to \infty$.*

Theorem 3.2.1. *The three types of convergence in the preceding definition are related as follows:* (1) *implies* (2) *and* (3) *if $\| x(\alpha) - x_n(\alpha) \|$ is measurable, and if $m(S) < \infty$ then* (2) *implies* (1) *and* (3); (3) *does not imply convergence anywhere.*

The proof goes as in the numerically-valued case and is omitted here.

DEFINITION 3.2.2. (1) $x(\alpha)$ *is said to be finitely-valued if it is constant on each of a finite number of disjoint measurable sets* S_j *with* $S = \bigcup S_j$. (2) *It is a simple function if it is finitely-valued and the set where* $x(\alpha) \neq \theta$ *is of finite measure.* (3) $x(\alpha)$ *is countably-valued if it assumes at most a countable set of values in* $\mathfrak{X}$ *each on a measurable set* S_j.

DEFINITION 3.2.3. $x(\alpha)$ *is said to be separably-valued if its range* $\mathfrak{R} = x(S)$ *is separable. It is almost separably-valued if there exists a null set* S_0 *in* S *such that* $x(S - S_0)$ *is separable.*

DEFINITION 3.2.4. (1) $x(\alpha)$ *is said to be weakly measurable in* S *if* $x^*[x(\alpha)]$ *is measurable* (Lebesgue) *in* S *for every* $x^* \in \mathfrak{X}^*$. (2) $x(\alpha)$ *is strongly measurable* (Bochner) *if there exists a sequence of countably-valued functions converging almost uniformly in* S *to* $x(\alpha)$.

If $m(S) < \infty$ we may replace "countably-valued" by "finitely-valued" in part (2) of the last definition. These various notions are connected by the following theorem due to B. J. Pettis [1].

THEOREM 3.2.2. $x(\alpha)$ *is strongly measurable in* S *if and only if it is weakly measurable and almost separably-valued.*

PROOF. We start with the necessity. If $x(\alpha)$ is strongly measurable, then there exists a null set S_0 in S and a sequence $\{x_n(\alpha)\}$ of countably-valued functions such that $\| x(\alpha) - x_n(\alpha) \| \to 0$ when $n \to \infty$ if $\alpha \in S - S_0$ (cf. Theorem 3.2.1). If $x^* \in \mathfrak{X}^*$, we have also $| x^*[x(\alpha)] - x^*[x_n(\alpha)] | \to 0$ in $S - S_0$. Here $x^*[x_n(\alpha)]$ takes on at most a countable set of distinct numerical values, each in a measurable set, and is consequently measurable (Lebesgue). Since $x^*[x(\alpha)]$ is the limit almost everywhere of a sequence of measurable functions, it is also measurable, so that $x(\alpha)$ is weakly measurable.

The values taken on by the functions $x_n(\alpha)$ form a countable set and the least closed linear manifold containing this set is consequently separable. By assumption $x(S - S_0)$ is a subset of this manifold and hence also separable. Thus $x(\alpha)$ is almost separably-valued.

We shall next prove that *if* $x(\alpha)$ *is weakly measurable and almost separably-valued, then* $\| x(\alpha) \|$ *is measurable* (Lebesgue). Without restricting the generality, we may assume that $x(\alpha)$ is actually separably-valued. We can then find a countable sequence $\{x_n\} = \{x(\alpha_n)\}$ dense in $x(S)$. Now let $x_n^* \in \mathfrak{X}^*$ be the functional such that $x_n^*(x_n) = \| x_n \|$ and $\| x_n^* \| = 1$. By assumption, $x_n^*[x(\alpha)]$ is measurable (Lebesgue) in S, so that $| x_n^*[x(\alpha)] |$ and $\sup_n | x_n^*[x(\alpha)] | \equiv f(\alpha)$ have the same property. Here $f(\alpha) \leqq \| x(\alpha) \|$ for all α and we have equality at least for $\alpha = \alpha_n$. However, for $\alpha \neq \alpha_n$ and any given $\epsilon > 0$ we may find an n such that $\| x(\alpha_n) \| > \| x(\alpha) \| - \epsilon$. We have then $f(\alpha) \geqq | x_n^*[x(\alpha)] | \geqq | x_n^*[x(\alpha_n)] | - | x_n^*[x(\alpha) - x(\alpha_n)] | > \| x(\alpha) \| - 2\epsilon$. Hence $f(\alpha) = \| x(\alpha) \|$ for all α and $\| x(\alpha) \|$ is measurable.

We shall now show that under the same assumptions $x(\alpha)$ may be approximated uniformly by countably-valued functions. We note first, by virtue of the result just proved, that each of the functions $\| x(\alpha) - x_n \|$ is measurable (Lebesgue). If m is a fixed positive integer, each of the sets S_{mn} where $\| x(\alpha) - x_n \| < 1/m$ is measurable and non-vacuous and $\bigcup_n S_{mn} = S$ since $\{x_n\}$ is dense in $x(S)$. We now set $S'_{m1} = S_{m1}$ and $S'_{mn} = S_{mn} - \bigcup_{j=1}^{n-1} S'_{mj}$ for $n > 1$. Then the sets S'_{mn} are disjoint, measurable, and their union is S. Defining $x_m(\alpha) = x_n$ in S'_{mn}, $m, n = 1, 2, 3, \cdots$, one sees that $x_m(\alpha)$ is a countably-valued function and $\| x(\alpha) - x_m(\alpha) \| < 1/m$ for all m and all α. Hence $x(\alpha)$ is the uniform limit of countably-valued functions and consequently strongly measurable. This completes the proof.

This theorem has interesting

Corollaries. (1) *If $\mathfrak{X}$ is separable, strong and weak measurability are equivalent notions.* (2) *A function $x(\alpha)$ is strongly measurable if and only if it is the uniform limit almost everywhere of a sequence of countably-valued functions.*

Strongly measurable functions have analogous properties to functions measurable (Lebesgue).

Theorem 3.2.3. (1) *If $x(\alpha)$ and $y(\alpha)$ are strongly measurable in S and γ_1, γ_2 are constants, then $\gamma_1 x(\alpha) + \gamma_2 y(\alpha)$ is strongly measurable.* (2) *If $f(\alpha)$ is a finite numerically-valued function which is measurable (Lebesgue), then $f(\alpha)x(\alpha)$ is strongly measurable if $x(\alpha)$ has this property.* (3) *If $x(\alpha)$ is the limit almost everywhere of a sequence of strongly measurable functions, then $x(\alpha)$ is strongly measurable.* (4) *The same conclusion is valid if in* (3) *the word "limit" (that is, strong limit) is replaced by "weak limit."* (5) *The conclusion is also valid if "limit almost everywhere" is replaced by "limit in measure."*

Proof. We shall give merely brief indications of the argument. (1) follows directly from Definition 3.2.4. (2) follows in the same manner if one keeps in mind that $f(\alpha)$ is the limit almost everywhere of countably-valued numerical functions. (3) If $x(\alpha)$ is the limit almost everywhere of the sequence $\{x_m(\alpha)\}$ of strongly measurable functions, then each $x_m(\alpha)$ is weakly measurable and almost separably-valued so that the limit $x(\alpha)$ has the same properties and is, hence, strongly measurable. (4) This follows from the fact that weak convergence of $x_n(\alpha)$ to $x(\alpha)$ (that is, convergence of $x^*[x_n(\alpha)]$ to $x^*[x(\alpha)]$ for all $x^* \in \mathfrak{X}^*$) implies strong convergence to $x(\alpha)$ of a sequence of suitably chosen linear combinations of the $x_n(\alpha)$. See S. Banach [**2**, p. 134]. (5) If $x_n(\alpha)$ converges in measure to $x(\alpha)$, then there exists a subsequence which converges strongly to $x(\alpha)$ almost everywhere.

Ordinarily $x(\alpha)y(\alpha)$ does not have a meaning in (B)-spaces, but if $\mathfrak{X}$ is a Banach algebra and not merely a Banach space, then the product is well defined and is strongly measurable whenever the factors have this property.

3.3. Operator functions and measure. The considerations of the preceding section apply also to the case in which $x(\alpha)$ is an element of a Banach algebra of

endomorphisms. But in this case a new set of conventions is more appropriate for the applications.

DEFINITION 3.3.1. *Let $U = U(\alpha)$ be a function on a set S in a k-dimensional euclidean space E_k to $\mathfrak{E}(\mathfrak{X})$, the Banach algebra of endomorphisms of a (B)-space $\mathfrak{X}$. Then $U(\alpha)$ will be called an operator function in $\mathfrak{X}$.*

DEFINITION 3.3.2. (1) *The operator function $U(\alpha)$ is said to be uniformly measurable in S if there exists a sequence of countably-valued operator functions in $\mathfrak{E}(\mathfrak{X})$ converging uniformly to $U(\alpha)$, uniformly with respect to α except in a null set.* (2) *$U(\alpha)$ is strongly measurable in S if the vector function $U(\alpha)[x]$ is strongly measurable in the sense of Definition* 3.2.4 (2) *for all $x \in \mathfrak{X}$.* (3) *$U(\alpha)$ is weakly measurable in S if $x^*\{U(\alpha)[x]\}$ is measurable (Lebesgue) for all $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$.*

The reader should observe that in part (1) the first and second "uniformly" refer to the topology while the third refers to α. In other words, there should exist a null set S_0, possibly void, a sequence $\{U_n(\alpha)\}$ of countably-valued functions in $\mathfrak{E}(\mathfrak{X})$, and to every $\epsilon > 0$ an $n(\epsilon)$ such that $\| U(\alpha) - U_n(\alpha) \| < \epsilon$ for $\alpha \in S - S_0$ and $n > n(\epsilon)$.

The connection between the three different types of measurability is given by the following theorem due to N. Dunford:

THEOREM 3.3.1. *A necessary and sufficient condition that $U(\alpha)$ be* (1) *strongly measurable is that $U(\alpha)$ be weakly measurable and that $U(\alpha)[x]$ be almost separably-valued in $\mathfrak{X}$ for all $x \in \mathfrak{X}$;* (2) *uniformly measurable is that $U(\alpha)$ be weakly measurable and almost separably-valued in $\mathfrak{E}(\mathfrak{X})$.*

PROOF. Part (1) is an immediate consequence of Theorem 3.2.2 and Definition 3.3.2 (2). The proof of part (2) follows the same lines as the proof of Theorem 3.2.2 and it is only in the proof of the measurability of $\| U(\alpha) \|$ that any modifications are necessary. It should be noted first that if $U(\alpha)$ is almost separably-valued in $\mathfrak{E}(\mathfrak{X})$, then $U(\alpha)[x]$ is almost separably-valued in $\mathfrak{X}$ for all x. Thus the first conclusion from the assumptions in part (2) is that $U(\alpha)$ is strongly measurable. In order to prove that $\| U(\alpha) \|$ is measurable we argue as follows. Without restricting the generality we may assume that $U(S)$ is separable. There is then a countable set $\{U_n\} = \{U(\alpha_n)\}$, dense in $U(S)$. To every n we can find a sequence $\{x_{mn}\}$ such that (i) $\| x_{mn} \| = 1$ and (ii) $\| U(\alpha_n) [x_{mn}] \| \geqq \| U(\alpha_n) \| - 1/m$. All the functions $\| U(\alpha)[x_{mn}] \|$ are measurable (Lebesgue) since $U(\alpha)$ is strongly measurable. Hence $F(\alpha) \equiv \sup_{m,n} \| U(\alpha) [x_{mn}] \|$ is also measurable and $F(\alpha) \leqq \| U(\alpha) \|$ for all α. Actually equality holds at least for $\alpha = \alpha_n$. However, for an $\alpha \neq \alpha_n$ we may choose an $n = n_m$ such that $\| U(\alpha) - U(\alpha_n) \| < 1/m$. Then

$$F(\alpha) \geqq \| U(\alpha)[x_{mn}] \| \geqq \| U(\alpha_n)[x_{mn}] \| - \| \{U(\alpha) - U(\alpha_n)\}[x_{mn}] \| > \| U(\alpha_n) \| - 2/m > \| U(\alpha) \| - 3/m$$

for every m. Hence $F(\alpha) = \| U(\alpha) \|$ for all α and $\| U(\alpha) \|$ is measurable. The proof is then completed as in Theorem 3.2.2.

3.4. Remarks on other properties. All function theory is based upon limiting processes, that is, ultimately upon a notion of convergence. In a (B)-space we have at least two such notions at our disposal, in a Banach algebra of endomorphisms at least three. This fact reflects itself in a corresponding multiplicity of all other concepts. This was illustrated above in the case of measurability. Other instances of importance for our later purposes are listed below for future reference.

DEFINITION 3.4.1. *A vector function $x(\alpha)$ on S to a (B)-space $\mathfrak{X}$ is* (1) *weakly continuous at $\alpha = \alpha_0$ if $\lim_{\alpha \to \alpha_0} | x^*[x(\alpha) - x(\alpha_0)] | = 0$ for all $x^* \in \mathfrak{X}^*$,* (2) *strongly continuous if $\lim_{\alpha \to \alpha_0} \| x(\alpha) - x(\alpha_0) \| = 0$.*

DEFINITION 3.4.2. *An operator function $U(\alpha)$ on S to $\mathfrak{E}(\mathfrak{X})$ is* (1) *weakly continuous at $\alpha = \alpha_0$ if $\lim_{\alpha \to \alpha_0} | x^*\{[U(\alpha) - U(\alpha_0)](x)\} | = 0$ for all $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$,* (2) *strongly continuous if $\lim_{\alpha \to \alpha_0} \| [U(\alpha) - U(\alpha_0)](x) \| = 0$ for all $x \in \mathfrak{X}$, and* (3) *uniformly continuous if $\lim_{\alpha \to \alpha_0} \| U(\alpha) - U(\alpha_0) \| = 0$.*

DEFINITION 3.4.3. *A vector function $x(\xi)$ on the interval (ξ_1, ξ_2) to the (B)-space $\mathfrak{X}$ is weakly (strongly) differentiable at $\xi = \xi_0$ if there is an element $x'(\xi_0) \in \mathfrak{X}$ such that the difference quotient $\alpha^{-1}[x(\xi_0 + \alpha) - x(\xi_0)]$ tends weakly (strongly) to $x'(\xi_0)$ when $\alpha \to 0$. We call $x'(\xi_0)$ the weak (strong) derivative of $x(\xi)$ at ξ_0.*

The extension to partial derivatives of functions of several variables is obvious. In the case of operator functions $U(\xi)$ we have a third possibility, that of *uniform differentiability* when $\| \alpha^{-1}[U(\xi_0 + \alpha) - U(\xi_0)] - U'(\xi_0) \| \to 0$ with α. *A weakly (strongly, uniformly) differentiable function is evidently weakly (strongly, uniformly) continuous.*

THEOREM 3.4.1. *If the weak derivative of $x(\xi)$ equals θ everywhere in (ξ_1, ξ_2), then $x(\xi)$ is a constant.*

PROOF. The assumption implies $dx^*[x(\xi)]/d\xi = 0$ and hence $x^*[x(\xi)] = x^*[x(\xi_0)]$ for all x^*, ξ_0 fixed in (ξ_1, ξ_2). By Theorem 2.10.4, $x(\xi) = x(\xi_0)$ for all ξ.

DEFINITION 3.4.4. *A vector function $x(\xi)$ on the closed interval $[\xi_1, \xi_2]$ to the (B)-space $\mathfrak{X}$ is of* (1) *weakly bounded variation in $[\xi_1, \xi_2]$ if $x^*[x(\xi)]$ is of bounded variation for every $x^* \in \mathfrak{X}^*$,* (2) *bounded variation if $\sup \| \sum [x(\beta_i) - x(\alpha_i)] \| < \infty$ for every choice of a finite number of non-overlapping intervals (α_i, β_i) in $[\xi_1, \xi_2]$, and* (3) *strongly bounded variation if $\sup \sum \| x(\alpha_i) - x(\alpha_{i-1}) \| < \infty$, where all possible partitions of $[\xi_1, \xi_2]$ are allowed. The two suprema are known as the total and the strong total variations respectively.*

THEOREM 3.4.2. *A function of weakly bounded variation is of bounded variation (but not necessarily of strongly bounded variation).*

For a proof see Theorem 12 of N. Dunford's paper [3].

Weak and strong forms of a function theoretical property are never independent. If the definitions are properly made, a stronger form always implies a weaker one. It sometimes happens that the converse is true so that the two forms are actually equivalent. This may be due either to the character of the space or to the character of the property in question. The first possibility is illustrated by Theorem 3.2.2 according to which weak and strong measurability coincide in separable spaces. An example of the second possibility is afforded by Theorem 3.4.2 above. Still another will be found in Theorem 3.9.1 below which asserts that all notions of holomorphism coincide for operator functions regardless of the character of the space. Here even the weakest form of the property in question requires such stringent cohesion between the values of the function that the strong and uniform conditions are satisfied automatically. A number of interesting instances in which a weaker property implies a strong form of the property has been found by N. Dunford (see the paper quoted above).

We list some conventions concerning infinite series with terms in a (B)-space. *Such a series will be said to converge strongly (weakly) to the sum s if the partial sums converge strongly (weakly) to s. The series $\sum u_n$ is said to be absolutely convergent if $\sum \| u_n \|$ converges*; an absolutely convergent series is obviously strongly convergent. If $u_n = u_n(\alpha)$, $\alpha \in S$, we have the possibility that the strong or weak convergence is uniform with respect to α; formal definitions are left to the reader. In the case of series of operator functions $\sum U_n(\alpha)$, great care should be taken in distinguishing between various types of uniform convergence. Such a series may be convergent in the uniform topology for fixed α or it may converge, in one sense or another, uniformly with respect to α. Both phenomena are properly referred to as uniform convergence, but the uniformity is in one case with respect to the elements on the unit sphere of the operand space $\mathfrak{X}$ and in the other with respect to a point variable in a parameter space.

3.5. Integration. The problem of developing a theory of abstract integration has attracted many authors during recent years. The Riemann integral was extended by L. M. Graves (1927); the Lebesgue integral by S. Bochner (1933), G. Birkhoff (1935), N. Dunford (1935–1938; three definitions, the D_0, D_1, and D_2 integrals mentioned below), I. Gelfand (1936–1938; two definitions G_1 and G_2 below), and B. J. Pettis (1938). The exact relations between the various extensions of the Lebesgue integral are not completely known at the present time. It is known, however, that the Bochner and the D_0 integrals, which are equivalent, are the most restrictive and that for strongly measurable functions the Birkhoff, the D_1, and the Pettis integrals are equivalent. The D_2 and the G_2 integrals are equivalent and the most inclusive definitions. Here the integral is no longer necessarily an element of the same space as the integrand but always belongs to the second adjoint space. We shall give explicit definitions of the Pettis and

the Bochner integrals, but devote most of the attention to the latter integral which is most important for our purposes. In the terminology of the preceding section we could describe the Pettis integral as a weak and the Bochner integral as a strong form of the concept "integral of an abstract function."

Definition 3.5.1. *The function $x(\alpha)$ on S to $\mathfrak{X}$ is integrable (Pettis) over S if and only if there is an element $x_S \in \mathfrak{X}$ such that for all $x^* \in \mathfrak{X}^*$*

$$\int_S x^*[x(\alpha)]\, d\alpha = x^*[x_S],$$

where the integral on the left is supposed to exist in the sense of Lebesgue. By definition,

$$(\mathrm{P}) \int_S x(\alpha)\, d\alpha = x_S .$$

Immediate consequences of the definition are: (1) *an integrable function is weakly measurable (not necessarily strongly measurable, however)*, (2) *the integral is uniquely defined*, (3) *it is a linear operation on $\mathfrak{X}$ to itself*, (4) *a simple function in the sense of Definition* 3.2.2 (2) *is integrable and*

$$(\mathrm{P}) \int_S x(\alpha)\, d\alpha = \sum x_j m(S_j),$$

(5) *if $\mathfrak{X}$ is the space of complex numbers, the definition coincides with the Lebesgue integral.*

A fundamental property of the integral is contained in

Theorem 3.5.1. *If T is a linear bounded transformation on the (B)-space $\mathfrak{X}$ to the (B)-space $\mathfrak{Y}$ with the same scalar field and if $x(\alpha) \in \mathfrak{X}$ is integrable (Pettis) over the set S, so is $T[x(\alpha)]$ and*

$$(\mathrm{P}) \int_S T[x(\alpha)]\, d\alpha = T[x_S].$$

Proof. This follows from the properties of the adjoint transformation T^*. It is required to show that $y^*\{T[x(\alpha)]\}$ is integrable (Lebesgue) for every $y^* \in \mathfrak{Y}^*$ and that the value of the integral is $y^*\{T[x_S]\}$. Now to a given $y^* \in \mathfrak{Y}^*$ corresponds a unique $x^* \in \mathfrak{X}^*$, $x^* = T^*(y^*)$, defined by $y^*[T(x)] = x^*(x)$, and

$$\int_S y^*\{T[x(\alpha)]\}\, d\alpha = \int_S x^*[x(\alpha)]\, d\alpha = x^*[x_S] = y^*\{T[x_S]\}$$

for every y^* as asserted.

Pettis observed that any definition of an integral of functions $x(\alpha)$ on S to $\mathfrak{X}$, which reduces to the definition of Lebesgue when $\mathfrak{X}$ is the space of complex numbers and has the property of Theorem 3.5.1, is included in his own in the

sense that the (P)-integral of $x(\alpha)$ must also exist and have the same value. To see this it is enough to take $\mathfrak{Y}$ equal to the space of complex numbers.

There are several alternate ways of introducing the Bochner integral. The following approach, which is due to Dunford, involves two steps.

Definition 3.5.2. *A countably-valued function $x(\alpha)$ on S to $\mathfrak{X}$ is integrable (Bochner) if and only if $\| x(\alpha) \|$ is integrable (Lebesgue). By definition*

$$(\mathrm{B}) \int_S x(\alpha)\, d\alpha = \sum_{j=1}^{\infty} x_j m(S_j).$$

The series converges since

$$\int_S \| x(\alpha) \| \, d\alpha = \sum_{j=1}^{\infty} \| x_j \| \, m(S_j).$$

Consequently

$$\left\| (\mathrm{B}) \int_S x(\alpha)\, d\alpha \right\| \leqq \int_S \| x(\alpha) \| \, d\alpha$$

for countably-valued functions. Further

$$\int_S x^*[x(\alpha)]\, d\alpha = \sum_{j=1}^{\infty} x^*(x_j) m(S_j)$$

for every $x^* \in \mathfrak{X}^*$, the series being absolutely convergent. It follows that the (B)- and the (P)-integrals of such functions coincide.

Definition 3.5.3. *A function $x(\alpha)$ on S to $\mathfrak{X}$ is integrable (Bochner) if and only if there exists a sequence of countably-valued functions $\{x_n(\alpha)\}$ converging almost uniformly to $x(\alpha)$ and such that*

$$\lim_{m,n\to\infty} \int_S \| x_m(\alpha) - x_n(\alpha) \| \, d\alpha = 0.$$

By definition

$$(\mathrm{B}) \int_S x(\alpha)\, d\alpha = \lim_{n\to\infty} (\mathrm{B}) \int_S x_n(\alpha)\, d\alpha.$$

We have to verify that the limit exists and is unique. The existence follows from the fact that the integrals in the right member have values in $\mathfrak{X}$ and form a Cauchy sequence in view of the inequality

$$\left\| \int_S x_m(\alpha)\, d\alpha - \int_S x_n(\alpha)\, d\alpha \right\| = \left\| \int_S [x_m(\alpha) - x_n(\alpha)]\, d\alpha \right\|$$
$$\leqq \int_S \| x_m(\alpha) - x_n(\alpha) \| \, d\alpha.$$

Here and in the following we omit the prefix (B) as long as the sense of the integral is clear from the context.

That the limit is independent of the defining sequence is shown by the following argument. Let $\{x_n(\alpha)\}$ be an admissible sequence, set $x_n = \int x_n(\alpha)d\alpha$, $x = \lim x_n$, and let $x^* \in \mathfrak{X}^*$ be an arbitrary linear functional. Then

$$x^*(x_n) = \int_S x^*[x_n(\alpha)]\, d\alpha \to x^*(x)$$

when $n \to \infty$. Since $x^*[x_n(\alpha)]$ converges in the mean of order one, and also converges almost uniformly to $x^*[x(\alpha)]$, it converges in the mean of order one to $x^*[x(\alpha)]$. This implies that

$$x^*(x) = \int_S x^*[x(\alpha)]\, d\alpha.$$

Now if $\{y_n(\alpha)\}$ is any other admissible sequence converging to $x(\alpha)$ and $y_n = \int y_n(\alpha)\, d\alpha$, $y = \lim y_n$, then by the same argument

$$x^*(y) = \int_S x^*[x(\alpha)]\, d\alpha.$$

Since $x^*(y) = x^*(x)$ for all $x^* \in \mathfrak{X}^*$, we have $y = x$ and the limit is unique.

THEOREM 3.5.2. *A necessary and sufficient condition that $x(\alpha)$ on S to $\mathfrak{X}$ be integrable (Bochner) is that $x(\alpha)$ be strongly measurable and that $\int_S \| x(\alpha) \|\, d\alpha < \infty$.*

PROOF. If $x(\alpha)$ is integrable then it is strongly measurable by virtue of Definitions 3.4.2 and 3.5.3. Hence $\| x(\alpha) \|$ is measurable (Lebesgue). From

$$\int_S \big| \| x_m(\alpha) \| - \| x_n(\alpha) \| \big|\, d\alpha \leqq \int_S \| x_m(\alpha) - x_n(\alpha) \|\, d\alpha \to 0$$

it follows that $\| x_n(\alpha) \|$, which tends almost uniformly to $\| x(\alpha) \|$, also converges in the mean of order one to $\| x(\alpha) \|$ so the latter function is integrable.

Conversely, if $x(\alpha)$ is strongly measurable and $\| x(\alpha) \|$ is integrable, then we may find a sequence of countably-valued functions $\{x_n(\alpha)\}$ converging almost uniformly to $x(\alpha)$ and satisfying the additional condition of converging in the mean of order one to $x(\alpha)$. If $m(S)$ is finite, the latter condition is implied by the first, otherwise it may be attained by a suitable modification of the original sequence. The existence of such a sequence $\{x_n(\alpha)\}$ implies the integrability of $x(\alpha)$.

We denote the class of Bochner integrable functions $x(\alpha)$ on S to $\mathfrak{X}$ by $B(S; \mathfrak{X})$.

THEOREM 3.5.3. *If* $x_1(\alpha)$ *and* $x_2(\alpha) \in B(S; \mathfrak{X})$ *and* γ_1, γ_2 *are constants, then* $\gamma_1 x_1(\alpha) + \gamma_2 x_2(\alpha) \in B(S; \mathfrak{X})$ *and*

$$\int_S [\gamma_1 x_1(\alpha) + \gamma_2 x_2(\alpha)]\, d\alpha = \gamma_1 \int_S x_1(\alpha)\, d\alpha + \gamma_2 \int_S x_2(\alpha)\, d\alpha.$$

PROOF. The theorem is true for countably-valued functions and follows for the general case by an obvious limiting process.

We prove similarly

THEOREM 3.5.4. *If* $x(\alpha) \in B(S; \mathfrak{X})$, *then*

$$\left\| \int_S x(\alpha)\, d\alpha \right\| \leqq \int_S \| x(\alpha) \|\, d\alpha.$$

THEOREM 3.5.5. *If* $S = \bigcup_n S_n$, $S_i \cap S_j = \emptyset$, *then*

$$\int_S x(\alpha)\, d\alpha = \sum_{n=1}^{\infty} \int_{S_n} x(\alpha)\, d\alpha.$$

In other words, *the integral is a completely additive function of sets.* The last two theorems together imply that *this set function is also absolutely continuous:*

THEOREM 3.5.6. *For every* $x(\alpha) \in B(S; \mathfrak{X})$ *and to every* $\epsilon > 0$ *there is a* $\delta = \delta(\epsilon, x)$ *such that for disjoint sets* S_n *in* S *the condition* $\sum_{n=1}^{\infty} m(S_n) < \delta$ *implies* $\sum_{n=1}^{\infty} \| \int_{S_n} x(\alpha)\, d\alpha \| < \epsilon$.

It is of fundamental importance for the applications that *Theorem* 3.5.1 *holds for* (*B*)*-integrals.* This is shown as follows. Suppose that $x(\alpha) \in B(S; \mathfrak{X})$ and that $y = T(x)$ is a linear bounded transformation on $\mathfrak{X}$ to $\mathfrak{Y}$. Let $\{x_n(\alpha)\}$ be countably-valued functions in $B(S; \mathfrak{X})$ such that $\int_S \| x(\alpha) - x_n(\alpha) \|\, d\alpha \to 0$. We note first that $T[x(\alpha)]$ belongs to $B(S; \mathfrak{Y})$. Indeed, if $x(\alpha)$ is almost separably-valued so is $T[x(\alpha)]$ and from $y^*\{T[x(\alpha)]\} = x^*[x(\alpha)]$, where $x^* = T^*(y^*)$, we conclude that weak measurability is preserved. Hence $T[x(\alpha)]$ is also strongly measurable and since $\| T[x(\alpha)] \| \leqq \| T \|\, \| x(\alpha) \|$, the norm $\| T[x(\alpha)] \|$ is integrable (Lebesgue) and finally $T[x(\alpha)]$ is (B)-integrable. The four expressions

$$T\left\{\int_S x(\alpha)\, d\alpha\right\}, \qquad \int_S T[x(\alpha)]\, d\alpha, \qquad T\left\{\int_S x_n(\alpha)\, d\alpha\right\}, \qquad \int_S T[x_n(\alpha)]\, d\alpha$$

then have a sense and define elements of $\mathfrak{Y}$. Here the third and the fourth expressions are obviously equal. Further the third expression converges to the first when $n \to \infty$ since T is continuous, and

$$\left\| \int_S T[x(\alpha)]\, d\alpha - \int_S T[x_n(\alpha)]\, d\alpha \right\| = \left\| \int_S T[x(\alpha) - x_n(\alpha)]\, d\alpha \right\|$$

$$\leqq \int_S \| T[x(\alpha) - x_n(\alpha)] \|\, d\alpha \leqq \| T \| \int_S \| x(\alpha) - x_n(\alpha) \|\, d\alpha \to 0.$$

This proves that

$$T\left\{\int_S x(\alpha)\ d\alpha\right\} = \int_S T[x(\alpha)]\,d\alpha \tag{3.5.1}$$

for (B)-integrable functions.

Since the Bochner integral clearly reduces to the Lebesgue integral when $\mathfrak{X}$ is the space of complex numbers and formula (3.5.1) holds, the remark of Pettis quoted above shows that *every (B)-integrable function is also (P)-integrable and the integrals have the same value.*

3.6. Further properties of the integral. We start with convergence in the mean and related questions.

THEOREM 3.6.1. *If $x_n(\alpha) \in B(S; \mathfrak{X})$ for all n and*

$$\lim_{m,n\to\infty} \int_S \| x_m(\alpha) - x_n(\alpha) \|\, d\alpha = 0,$$

then there exists an element $x(\alpha) \in B(S; \mathfrak{X})$ such that

$$\lim_{n\to\infty} \int_S \| x(\alpha) - x_n(\alpha) \|\, d\alpha = 0.$$

If $y(\alpha)$ has the same property, then $x(\alpha) = y(\alpha)$ almost everywhere.

PROOF. Brief indications will suffice since the argument follows closely the classical proof. We select a subsequence $\{x_{n_j}(\alpha)\}$ subject to the condition

$$\int_S \| x_{n_j}(\alpha) - x_n(\alpha) \|\, d\alpha < 2^{-j} \qquad \text{for } n > n_j.$$

The series

$$x_{n_1}(\alpha) + \sum_2^\infty [x_{n_j}(\alpha) - x_{n_{j-1}}(\alpha)]$$

converges for almost all α since the integral of the sum of the norms converges. The sum $x(\alpha)$ is strongly measurable by Theorem 3.2.3 (3) and $\| x(\alpha) \|$ is integrable. Hence $x(\alpha) \in B(S; \mathfrak{X})$. For fixed n the function $\| x_{n_j}(\alpha) - x_n(\alpha) \|$ is dominated by a fixed integrable function for all j and $\| x_{n_j}(\alpha) - x_n(\alpha) \| \to \| x(\alpha) - x_n(\alpha) \|$ for almost all α when $j \to \infty$. Hence

$$\int_S \| x(\alpha) - x_n(\alpha) \|\, d\alpha \leqq 2^{-j} \qquad \text{for } n > n_j.$$

The uniqueness almost everywhere of the limit is proved by the usual argument.

In other words, *a sequence $\{x_n(\alpha)\}$ in $B(S; \mathfrak{X})$ which converges in the mean of order one converges in the mean of order one to a limit in $B(S; \mathfrak{X})$.* This result implies

THEOREM 3.6.2. *The set $B(S; \mathfrak{X})$ becomes a (B)-space if the norm of the element $x(\cdot)$ is defined to be*

$$\| x(\cdot) \| = \int_S \| x(\alpha) \| \, d\alpha.$$

Definition 3.5.3 shows that *countably-valued functions are dense in $B(S; \mathfrak{X})$* and this implies in turn that *simple functions are also dense.* This means that *the two-valued simple functions* $[x(\alpha) = a$ on S_1, θ on $S - S_1]$ *form a fundamental set in $B(S; \mathfrak{X})$.* This set may be reduced, however, provided S is a connected convex set as we may assume without restricting the generality. A classical argument shows that it is sufficient to restrict S_1 to be an oriented k-dimensional interval, I: $(\gamma_{11} < \alpha_1 < \gamma_{12}, \cdots, \gamma_{k1} < \alpha_k < \gamma_{k2})$. The corresponding functions $x(\alpha) = a$ on I, θ on $S - I$ form a fundamental set. Such a step function can obviously be approximated in the mean arbitrarily closely by continuous functions. Hence *the continuous functions are also dense in $B(S; \mathfrak{X})$.* From this fact we conclude that *the functions of $B(S; \mathfrak{X})$ are continuous in the mean.* To simplify the formulation we take $S = E_k$.

THEOREM 3.6.3. *If $x(\alpha) \in B(E_k; \mathfrak{X})$, then*

$$\lim_{\eta \to 0} \int_{E_k} \| x(\alpha + \mathfrak{n}) - x(\alpha) \| \, d\alpha = 0.$$

Here $x(\alpha + \mathfrak{n}) = x(\alpha_1 + \eta_1, \cdots, \alpha_k + \eta_k)$. An important consequence in

THEOREM 3.6.4. *If $x(\alpha) \in B(E_k; \mathfrak{X})$ and $f(\alpha)$ is a bounded numerically-valued measurable function, then*

$$y(\xi) = \int_{E_k} f(\alpha) x(\alpha + \xi) \, d\alpha$$

is a continuous function of ξ.

PROOF. The integral obviously exists and defines an element of $\mathfrak{X}$. If $| f(\alpha) | \leqq M$ we have

$$\| y(\xi + \mathfrak{n}) - y(\xi) \| \leqq \int_{E_k} | f(\alpha) | \, \| x(\alpha + \xi + \mathfrak{n}) - x(\alpha + \xi) \| \, d\alpha$$

$$\leqq M \int_{E_k} \| x(\beta + \mathfrak{n}) - x(\beta) \| \, d\beta \to 0 \text{ with } \mathfrak{n}.$$

In addition to the class $B(S; \mathfrak{X}) = B_1(S; \mathfrak{X})$, mention should be made of the classes $B_p(S; \mathfrak{X})$, $1 < p < \infty$. *A function $x(\alpha)$ on S to $\mathfrak{X}$ belongs to $B_p(S; \mathfrak{X})$ if $x(\alpha)$ is strongly measurable in S and* $\int_S \| x(\alpha) \|^p \, d\alpha < \infty$. Similarly $x(\alpha) \in B_\infty(S; \mathfrak{X})$, *if $x(\alpha)$ is strongly measurable and $\| x(\alpha) \|$ is bounded except in a null set. The class $B_p(S; \mathfrak{X})$ becomes a (B)-space under the norm*

$$\| x(\cdot) \|_p = \left\{ \int_S \| x(\alpha) \|^p \, d\alpha \right\}^{1/p}, \qquad \| x(\cdot) \|_\infty = \text{ess. sup} \, \| x(\alpha) \|.$$

The obvious generalizations of Theorems 3.6.1, 3.6.3, and 3.6.4 also are valid. In the last theorem we have now to suppose that $f(\alpha) \in L_{p'}(E_k)$ where $1/p + 1/p' = 1$. The analogy between the spaces $L_p(S)$ and $B_p(S; \mathfrak{X})$ is not complete, however; in particular, classical Fourier analysis does not carry over without serious impairment.

We return to $B(E_k ; \mathfrak{X})$. The question of differentiability of the indefinite integral is of paramount importance. Let $x(\alpha) \in B(E_k ; \mathfrak{X})$, let $C(\xi, \gamma)$ be the cube

$$\xi_1 - \gamma < \alpha_1 < \xi_1 + \gamma, \cdots, \xi_k - \gamma < \alpha_k < \xi_k + \gamma,$$

and put

$$\varphi(x; \xi, \gamma) = (2\gamma)^{-k} \int_{C(\xi,\gamma)} x(\alpha)\, d\alpha.$$

THEOREM 3.6.5. *For almost all* ξ

$$\lim_{\gamma\to 0} \varphi(x; \xi, \gamma) = x(\xi).$$

PROOF. The theorem is obviously true for continuous functions. Let $\{x_n(\alpha)\}$ be a sequence of continuous functions in $B(E_k ; \mathfrak{X})$ tending to $x(\alpha)$ for almost all α. For fixed n

$$\limsup_{\gamma\to 0} \| \varphi(x; \xi, \gamma) - x(\xi) \| \leqq \limsup_{\gamma\to 0} \varphi(\| x - x_n \|; \xi, \gamma)$$
$$+ \limsup_{\gamma\to 0} \| \varphi(x_n ; \xi, \gamma) - x_n(\xi) \| + \| x(\xi) - x_n(\xi) \|.$$

The first term on the right equals $\| x(\xi) - x_n(\xi) \|$ for almost all ξ by the classical theorem of Lebesgue and the second term is zero for all ξ since $x_n(\xi)$ is continuous. Hence for all n and almost all ξ

$$\limsup_{\gamma\to 0} \| \varphi(x; \xi, \gamma) - x(\xi) \| \leqq 2 \| x(\xi) - x_n(\xi) \|.$$

But when $n \to \infty$, the right-hand side tends to zero for almost all ξ. Hence

$$\limsup_{\gamma\to 0} \| \varphi(x; \xi, \gamma) - x(\xi) \| = 0$$

for almost all ξ and the theorem is proved.

In this theorem we may replace the cube $C(\xi, \gamma)$ by certain other measurable point sets $S(\xi, \gamma)$ which shrink to the point ξ when $\gamma \to 0$. The quantity $(2\gamma)^k$ is then to be replaced by $m[S(\xi, \gamma)]$. If $k = 1$, we may take in particular either of the intervals $(\xi - \gamma, \xi)$ or $(\xi, \xi + \gamma)$. This leads to the following

COROLLARY. *If* $x(\alpha) \in B(E_1 ; \mathfrak{X})$, *then for almost all* ξ

$$\lim_{\gamma\to 0} \frac{1}{\gamma} \int_{\xi}^{\xi+\gamma} x(\alpha)\, d\alpha = x(\xi).$$

It is also possible to prove the sharper statement:

$$\lim_{\gamma \to 0} \frac{1}{\gamma} \int_{\xi}^{\xi+\gamma} \| x(\alpha) - x(\xi) \| \, d\alpha = 0 \tag{3.6.1}$$

for almost all ξ. For a proof, see pp. 271–272 of Bochner's article.

The classical theorem of Lebesgue on passage to the limit under the sign of integration holds for (B)-integrals.

THEOREM 3.6.6. *If* $x_n(\alpha) \in B(S; \mathfrak{X})$ *for all* n *and converges almost uniformly to a limit function* $x(\alpha)$ *and if there exists a fixed function* $F(\alpha) \in L(S)$ *such that* $\| x_n(\alpha) \| \leqq F(\alpha)$ *for all* n *and* α, *then* $x(\alpha) \in B(S; \mathfrak{X})$ *and*

$$\lim_{n \to \infty} \int_S x_n(\alpha) \, d\alpha = \int_S x(\alpha) \, d\alpha.$$

The proof may be left to the reader. We see in particular that *the conclusion is valid if* $x_n(\alpha)$ *converges boundedly to* $x(\alpha)$ *and the measure of* S *is finite.* Another sufficient condition is that $x_n(\alpha)$ *converges in the mean of order one to* $x(\alpha)$.

Finally we mention that *the Fubini theorem holds for (B)-integrals.* We shall make occasional use of this theorem in the following formulation:

THEOREM 3.6.7. *If* $x(\alpha, \beta)$ *is a strongly measurable function of* $(\alpha, \beta) = (\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n)$, *then* $x(\alpha, \beta) \in B(E_{m+n}; \mathfrak{X})$ *if there is a function* $y(\alpha, \beta) = x(\alpha, \beta)$ *for almost all* (α, β) *such that*

$$\int_{E_n} \left\{ \int_{E_m} \| y(\alpha, \beta) \| \, d\alpha \right\} d\beta$$

exists. In this case

$$\begin{aligned} \int_{E_{m+n}} x(\alpha, \beta) \, d\alpha \, d\beta &= \int_{E_{m+n}} y(\alpha, \beta) \, d\alpha \, d\beta \\ &= \int_{E_n} \left\{ \int_{E_m} y(\alpha, \beta) \, d\alpha \right\} d\beta = \int_{E_m} \left\{ \int_{E_n} y(\alpha, \beta) \, d\beta \right\} d\alpha. \end{aligned}$$

3.7. Singular integrals. A considerable body of the classical theory of singular integrals carries over to the case of vector-valued functions. We shall state several theorems concerning such integrals with brief indications of the proofs which follow standard patterns. Throughout the discussion the *kernel* $K(\xi, \alpha; \omega)$ is a numerically-valued function defined for $-\infty < \xi, \alpha < \infty, \omega > 0$, measurable in (ξ, α) as well as in ξ and α separately for all values of the other variable. We set

$$x(\xi; \omega) = \int_{E_1} K(\xi, \alpha; \omega) x(\alpha) \, d\alpha, \tag{3.7.1}$$

where $x(\alpha)$ is vector-valued and the existence of the integral will be ensured by further assumptions.

THEOREM 3.7.1. *Let $K(\xi, \alpha; \omega)$ satisfy the conditions:*

(1) *$K(\xi, \alpha; \omega) \in L_1(E_1)$ as a function of ξ for all α and as a function of α for all ξ when ω is fixed but arbitrary;*

(2) *$\int_{E_1} | K(\xi, \alpha; \omega) | d\xi \leqq A$ for all α and ω;*

(3) *$\lim_{\omega\to\infty} \int_{E_1 - I} | K(\xi, \alpha; \omega) | d\xi = 0$ for every open interval I containing α;*

(4) *$\lim_{\omega\to\infty} \int_I K(\xi, \alpha; \omega)\, d\alpha = 1$ for every open interval I containing ξ.*

If $x(\alpha) \in B_1(E_1 ; \mathfrak{X})$, then

(i) *$x(\xi; \omega)$ exists for almost all ξ and belongs to $B_1(E_1 ; \mathfrak{X})$;*

(ii) *$\| x(\cdot\, ; \omega) \|_1 \leqq A \| x(\cdot) \|_1$;*

(iii) *$\lim_{\omega\to\infty} \| x(\cdot) - x(\cdot\, ; \omega) \|_1 = 0$.*

PROOF. The measurability assumptions ensure that the integrand in (3.7.1) is a strongly measurable function of α for all ξ and ω. That $| K(\xi, \alpha; \omega) | \, \| x(\alpha) \|$ is integrable follows from the Fubini theorem by virtue of the inequality

$$\int_{E_1} \left\{ \int_{E_1} | K(\xi, \alpha; \omega) | \, \| x(\alpha) \| \, d\xi \right\} d\alpha \leqq A \, \| x(\cdot) \|_1$$

which is implied by (1) and (2). This proves (i) and (ii). If $x(\alpha)$ is the step function $x_1(\alpha) = a$ in I and θ outside, $m(I) < \infty$, then

$$\| x_1(\cdot) - x_1(\cdot\, ; \omega) \|_1 = \| a \| \int_I \left| \int_I K(\xi, \alpha; \omega)\, d\alpha - 1 \right| d\xi + \| a \| \int_{E_1 - I} \left| \int_I K(\xi, \alpha; \omega)\, d\alpha \right| d\xi.$$

Both terms on the right tend to zero when $\omega \to \infty$ by virtue of conditions (4) and (3). But these step functions form a fundamental set in $B_1(E_1 ; \mathfrak{X})$ and the operation which takes $x(\cdot)$ into $x(\cdot\, ; \omega)$ is linear and bounded uniformly with respect to ω. The Banach-Steinhaus theorem 2.12.1 then shows that (iii) holds for every $x(\cdot) \in B_1(E_1 ; \mathfrak{X})$.

The conditions of Theorem 3.7.1 are not suitable for a discussion of pointwise convergence. The choice of additional conditions is simplest in the case of convergence at points of continuity.

THEOREM 3.7.2. *Let $K(\xi, \alpha; \omega)$ satisfy conditions (1) and (4) of the preceding theorem and in addition:*

(5) *to every ξ there is a finite $M_1(\xi, \omega)$ such that $| K(\xi, \alpha; \omega) | \leqq M_1(\xi, \omega)$ for all α;*

(6) *to every ξ and every $\epsilon > 0$ there is a finite $M_2(\xi, \epsilon)$ such that $| K(\xi, \alpha; \omega) | < M_2(\xi, \epsilon)$ for all ω when α is outside of $(\xi - \epsilon, \xi + \epsilon)$;*

(7) *to every ξ there is a finite $M(\xi)$ such that $\int_{\xi-1}^{\xi+1} | K(\xi, \alpha; \omega) | \, d\alpha \leqq M(\xi)$ for all ω.*

If $x(\alpha) \in B_1(E_1 ; \mathfrak{X})$ then $x(\xi; \omega)$ exists for all ξ and

$$\lim_{\omega\to\infty} x(\xi;\ \omega) = x(\xi)$$

at all points of continuity of $x(\alpha)$.

Proof. The existence of $x(\xi; \omega)$ follows from the assumptions of measurability together with (5). We note next that (4) implies that

$$\lim_{\omega\to\infty} \int_I K(\xi, \alpha; \omega)\, d\alpha = 0$$

for every closed interval I which does not contain the point ξ.

Let ξ be a point of continuity of $x(\alpha)$ and break up the integral defining $x(\xi; \omega)$ into three parts, J_1, J_2, and J_3, using $\alpha = \xi - \epsilon$ and $\xi + \epsilon$ as partition points. Here $\epsilon = \epsilon(\delta)$ is chosen so small that $\| x(\alpha) - x(\xi) \| \leqq \delta$ when $| \alpha - \xi | \leqq \epsilon$. Then

$$J_2 - x(\xi) = \left\{ \int_{\xi-\epsilon}^{\xi+\epsilon} K(\xi, \alpha; \omega)\, d\alpha - 1 \right\} x(\xi) + \int_{\xi-\epsilon}^{\xi+\epsilon} K(\xi, \alpha; \omega)[x(\alpha) - x(\xi)]\, d\alpha.$$

The first term on the right tends to zero when $\omega \to \infty$ by (4) and the norm of the second term is less than $\delta M(\xi)$ by (7). Thus $\limsup_{\omega\to\infty} \| J_2 - x(\xi) \| \leqq \delta M(\xi)$.

If $x(\alpha)$ is a step function, we have

$$J_1 + J_3 = \sum_1^n a_j \int_{I_j} K(\xi, \alpha; \omega)\, d\alpha,$$

and, since ξ lies outside of all the intervals I_j, each integral tends to zero when $\omega \to \infty$. In the general case we may approximate $x(\alpha)$ in the mean of order one by a step function $x_0(\alpha)$ such that $\| x(\cdot) - x_0(\cdot) \|_1 < \eta$. We have then, with obvious notation, $\| J_1 + J_3 - J_{10} - J_{30} \| < \eta M_2(\xi, \epsilon)$ by (6) and consequently

$$\limsup_{\omega\to\infty} \| x(\xi) - x(\xi; \omega) \| \leqq \delta M(\xi) + \eta M_2(\xi, \epsilon).$$

Here δ and η are arbitrary whence it follows that the right-hand side may be replaced by zero. This completes the proof.

Remark. *Suppose that condition* (4) *is satisfied in the following stronger form:* (4′) *For every* $\delta > 0$ *and all* ξ

$$\lim_{\omega\to\infty} \int_{\xi-\delta}^{\xi} K(\xi, \alpha; \omega)\, d\alpha = \mu_1, \qquad \lim_{\omega\to\infty} \int_{\xi}^{\xi+\delta} K(\xi, \alpha; \omega)\, d\alpha = \mu_2,$$

where $\mu_1 + \mu_2 = 1$ *and* μ_1, μ_2 *are independent of* ξ.

Then, under the assumptions (1), (4′), (5), (6), *and* (7),

$$\lim_{\omega\to\infty} x(\xi; \omega) = \mu_1 x(\xi - 0) + \mu_2 x(\xi + 0)$$

at all points where $x(\xi)$ *has left- and right-hand limits.*

This is proved by the same methods as the preceding theorem.

Assumptions of a more special nature have to be made to attain convergence almost everywhere.

THEOREM 3.7.3. *Let $K(\xi, \alpha; \omega)$ satisfy conditions* (1), (4), *and* (6) *and, in addition*:

(8) *there exists a non-negative function $P(\beta, \omega)$ such that*

(i) $|K(\xi, \alpha; \omega)| \leqq P(|\alpha - \xi|; \omega)$ *for all ω and all α, ξ with* $|\alpha - \xi| \leqq 1$;

(ii) $P(\beta; \omega)$ *is a bounded decreasing function of β for fixed ω*;

(iii) $\int_0^1 P(\beta; \omega)\, d\beta \leqq M$ *for all* $\omega > 0$.

If $x(\alpha) \in B_1(E_1; \mathfrak{X})$ then $x(\xi; \omega)$ exists for all ξ and $\lim_{\omega\to\infty} x(\xi; \omega) = x(\xi)$ almost everywhere, in particular, in the Lebesgue set of $x(\alpha)$ where formula (3.6.1) *is valid.*

PROOF. The existence of $x(\xi; \omega)$ for all ξ follows from the measurability together with conditions (6) and (8ii). Now let ξ be a point where (3.6.1) is valid and break up the integral defining $x(\xi; \omega)$ as in the preceding proof. The discussion of $J_1 + J_3$ which is based on (4) and (6) goes as before. In the discussion of J_2 we note that

$$\left\| \int_{\xi-\epsilon}^{\xi+\epsilon} K(\xi, \alpha; \omega)[x(\alpha) - x(\xi)]\, d\alpha \right\| < \int_{-\epsilon}^{\epsilon} P(|\beta|; \omega)\, \| x(\xi + \beta) - x(\xi) \|\, d\beta \equiv J_4.$$

If now

$$X(\beta; \xi) = \int_0^{\beta} \| x(\xi + \gamma) - x(\xi) \|\, d\gamma,$$

then $|X(\beta; \xi)| < \delta |\beta|$ if $|\beta| \leqq \epsilon = \epsilon(\delta)$. An integration by parts gives

$$J_4 = [X(\epsilon; \xi) - X(-\epsilon; \xi)]P(\epsilon; \omega) - \int_{-\epsilon}^{\epsilon} X(\beta; \xi)\, d_\beta P(|\beta|; \omega)$$

$$< 2\delta \left[\epsilon P(\epsilon; \omega) - \int_0^{\epsilon} \beta\, d_\beta P(\beta; \omega) \right] = 2\delta \int_0^{\epsilon} P(\beta; \omega)\, d\beta < 2\delta M,$$

if $\epsilon < 1$ as we may assume. The proof is then completed as before.

These three theorems are typical for the theory and the reader will have no difficulties in proving analogous theorems for other classes of functions.

3.8. Riemann-Stieltjes integrals. In extending the notion of a Stieltjes integral to vector-valued functions we have two possibilities: *either the integrand or the integrator may be vector-valued.* Both varieties will occur in the following, but we may restrict ourselves to the Riemann-Stieltjes type of integral.

THEOREM 3.8.1. *Let $x(\xi)$ be a strongly continuous function on the interval $[\xi_1, \xi_2]$ to $\mathfrak{X}$ and let $g(\xi)$ be a numerically-valued function of bounded variation in the same interval. Then*

$$\int_{\xi_1}^{\xi_2} x(\xi)\, dg(\xi)$$

exists as the unique strong limit of Riemann sums of the form

$$\sum x(\tau_i)[g(\sigma_i) - g(\sigma_{i-1})].$$

Further, if T is a linear bounded transformation on $\mathfrak{X}$ to $\mathfrak{Y}$ then

$$T\left\{\int_{\xi_1}^{\xi_2} x(\xi)\, dg(\xi)\right\} = \int_{\xi_1}^{\xi_2} T[x(\xi)]\, dg(\xi).$$

Proof. If the points $\{\alpha_i\}$ and $\{\beta_i\}$ define two partitions of $[\xi_1, \xi_2]$ such that $\| x(\xi') - x(\xi'') \| \leqq \epsilon$ for ξ' and ξ'' in the same α- or β-interval and if S_1 and S_2 are two Riemann sums corresponding to these partitions, then a simple calculation shows that

$$\| S_1 - S_2 \| \leqq 2\epsilon \int_{\xi_1}^{\xi_2} | dg(\xi) |.$$

This proves the existence of the integral. The second part follows from the linearity and continuity of T by applying T to a sequence of approximating Riemann sums.

Theorem 3.8.2. *Let $f(\xi)$ be a continuous numerically-valued function in $[\xi_1, \xi_2]$ and $x(\xi)$ a function on $[\xi_1, \xi_2]$ to $\mathfrak{X}$ which is of bounded variation in the sense of Definition* 3.4.4 (2). *Then*

$$\int_{\xi_1}^{\xi_2} f(\xi)\, dx(\xi)$$

exists as the unique strong limit of Riemann sums. Further, if T is a linear bounded transformation on $\mathfrak{X}$ to $\mathfrak{Y}$, then

$$T\left\{\int_{\xi_1}^{\xi_2} f(\xi)\, dx(\xi)\right\} = \int_{\xi_1}^{\xi_2} f(\xi)\, d[Tx(\xi)].$$

Proof. For the existence of the integral, see N. Dunford [3, Theorem 11]. For the second part we note that $x(\xi)$ being of bounded variation implies that $T[x(\xi)]$ has the same property and the total variation of $T[x(\xi)]$ does not exceed $\| T \|$ times the total variation of $x(\xi)$. The desired relation then follows by linearity and continuity from the corresponding relations for the approximating Riemann sums.

References. Banach [2], G. Birkhoff [1], Bochner [2], Dunford [1, 2, 3], Gelfand [1, 2], Graves [1], and Pettis [1].

2. Complex Function Theory

3.9. Holomorphic functions. The theory of analytic functions on the complex plane to a linear vector space goes back to D. Hilbert [1] and F. Riesz [2]. For the case of a general (B)-space the basic extensions are due to Norbert Wiener

[1]. In recent years N. Dunford, L. Fantappiè, I. Gelfand, E. R. Lorch, M. H. Stone, and A. E. Taylor have done much to broaden our knowledge in this field.

The basic concepts are those of a *holomorphic vector function* and a *holomorphic operator function*. Let D be a domain of the complex ζ-plane, $\zeta = \xi + i\eta$, and let $x(\zeta)$ be a function on D to a complex (B)-space $\mathfrak{X}$, $U(\zeta)$ a function on D to $\mathfrak{E}(\mathfrak{X})$. In classical function theory a function $f(\zeta)$ is holomorphic in the domain D if it is single-valued, continuous, and differentiable. The first of these notions carries over to abstract functions without ambiguity, but the second and third notions have two different meanings for vector functions and three for operator functions depending upon which notion of convergence we use. Nevertheless we arrive at a unique concept of holomorphism.

DEFINITION 3.9.1. *$x(\zeta)$ and $U(\zeta)$ are said to be holomorphic in D if $x^*[x(\zeta)]$ and $x^*\{U(\zeta)[x]\}$ are holomorphic in Cauchy's sense for every choice of $x^* \in \mathfrak{X}^*$ and $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$ respectively.*

This is the weakest possible assumption and still it implies the strongest possible conclusion. For vector functions this was first proved by N. Dunford [3, p. 354] and for operator functions by E. Hille [7, p. 6]. See also A. E. Taylor ([5, p. 576] and [7, p. 653]). The proofs are beautiful applications of the principle of uniform boundedness.

THEOREM 3.9.1. (1) *If $x(\zeta)$ is holomorphic in D, then $x(\zeta)$ is strongly continuous and strongly differentiable in D, uniformly with respect to ζ in any domain D_0 which is bounded and strictly interior to D.* (2) *If $U(\zeta)$ is holomorphic in D, then $U(\zeta)$ is uniformly continuous and uniformly differentiable in D, uniformly with respect to ζ in D_0.*

PROOF. It is enough to prove (2) which exhibits the method. We base the proof upon the fact that *the difference quotient of a numerically-valued function $f(\zeta)$ which is holomorphic in D tends to its limit, the derivative, uniformly with respect to ζ in any domain D_0 which is bounded and has a positive distance from the boundary of D.* Expressed as an inequality for difference quotients we may formulate this observation as follows:

LEMMA 3.9.1. *To any function $f(\zeta)$ holomorphic in the domain D and any domain D_0 which is bounded and strictly interior to D there is a finite quantity $M(f; D_0)$ such that for every choice of ζ, $\zeta + \alpha$, and $\zeta + \beta$ in D_0*

$$\left|\frac{1}{\alpha - \beta}\left\{\frac{1}{\alpha}[f(\zeta + \alpha) - f(\zeta)] - \frac{1}{\beta}[f(\zeta + \beta) - f(\zeta)]\right\}\right| \leqq M(f; D_0).$$

This follows from the fact that the function inside the absolute value signs on the left is represented by the Cauchy integral

$$\frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t - \zeta)(t - \zeta - \alpha)(t - \zeta - \beta)},$$

where C consists of a finite number of closed simple rectifiable curves in D having a positive minimal distance both from D_0 and from the boundary of D. The conclusion is immediate.

We apply this Lemma to the function $x^*\{U(\zeta)[x]\}$ where x and x^* are arbitrary elements of $\mathfrak{X}$ and $\mathfrak{X}^*$ respectively. To simplify the notation somewhat we write

$$\frac{1}{\alpha - \beta}\left\{\frac{1}{\alpha}[U(\zeta + \alpha) - U(\zeta)] - \frac{1}{\beta}[U(\zeta + \beta) - U(\zeta)]\right\} = U(\zeta; \alpha, \beta).$$

The lemma then asserts that

$$| x^*\{U(\zeta; \alpha, \beta)[x]\} | \leqq M(x^*, x, U; D_0)$$

for every choice of ζ, $\zeta + \alpha$ and $\zeta + \beta$ in D_0. By Theorem 2.12.3 this implies the existence of a finite quantity $M(x, U; D_0)$ such that

$$|| U(\zeta; \alpha, \beta)[x] || \leqq M(x, U; D_0).$$

But now Theorem 2.12.2 applies and ensures the existence of a finite $M(U; D_0)$ such that

$$|| U(\zeta; \alpha, \beta) || \leqq M(U; D_0).$$

Here we let $\beta \to 0$ and use the fact that $\mathfrak{E}(\mathfrak{X})$ is complete. Since the difference quotient tends to a limit, there exists a linear bounded operator function $U'(\zeta) \in \mathfrak{E}(\mathfrak{X})$ such that

$$\left\| \frac{1}{\alpha}[U(\zeta + \alpha) - U(\zeta)] - U'(\zeta) \right\| \leqq | \alpha | M(U; D_0)$$

for all ζ and $\zeta + \alpha$ in D_0. Thus $U(\zeta)$ has a uniform derivative in D and the difference quotient approaches the derivative uniformly with respect to ζ in D_0. This of course implies uniform continuity as well as strong and weak differentiability and continuity.

It should be observed that the assumption $U(\zeta) \in \mathfrak{E}(\mathfrak{X})$ is not essential; the proof applies just as well to a linear bounded operator function on one (B)-space to another.

3.10. Cauchy's integral. Let C be a rectifiable curve in the complex ζ-plane given by the equation $\zeta = \zeta(\alpha)$, $0 \leqq \alpha \leqq \alpha_0$, where $\zeta(\alpha)$ is continuous and of bounded variation in $[0, \alpha_0]$. If $x(\zeta)$ is any strongly continuous function on C to a (B)-space $\mathfrak{X}$, then the integral

$$\int_0^{\alpha_0} x[\zeta(\alpha)]\, d\zeta(\alpha) \equiv \int_C x(\zeta)\, d\zeta$$

exists by Theorem 3.8.1 and if T is any linear bounded transformation on $\mathfrak{X}$ to $\mathfrak{Y}$

$$T\left\{\int_C x(\zeta)\, d\zeta\right\} = \int_C T[x(\zeta)]\, d\zeta.$$

Similar formulas hold if $x(\zeta)$ is replaced by a uniformly continuous operator function on C to $\mathfrak{E}(\mathfrak{X})$, but in this case we have also relations of the type

$$\left\{\int_C U(\zeta)\,d\zeta\right\} x = \int_C U(\zeta)[x]\,d\zeta$$

and

$$T\left\{\int_C U(\zeta)[x]\,d\zeta\right\} = \int_C T\{U(\zeta)[x]\}\,d\zeta.$$

One may verify both formulas by observing that corresponding relations hold for the approximating Riemann-Stieltjes sums which tend to the integrals in question, strongly with respect to the $\mathfrak{X}$- and $\mathfrak{Y}$-metrics respectively.

The restrictive assumption that $x(\zeta)$ and $U(\zeta)$ be continuous is a matter of convenience and simplicity and may be replaced by measurability and integrability of the norm with respect to the length of arc.

So far the functions $x(\zeta)$ and $U(\zeta)$ were arbitrary except for continuity assumptions. Suppose now that they are holomorphic in a domain D of the ζ-plane and that C is a simple closed rectifiable curve in D, the interior of C being also in D. We now take $\mathfrak{Y}$ as the space of complex numbers and $T = x^*$ an arbitrary element of $\mathfrak{X}^*$. It follows that

$$x^*\left\{\int_C U(\zeta)[x]\,d\zeta\right\} = \int_C x^*\{U(\zeta)[x]\}\,d\zeta = 0$$

for every x^*. By Theorem 2.10.4 this requires that

$$\int_C U(\zeta)[x]\,d\zeta = \left\{\int_C U(\zeta)\,d\zeta\right\}[x] = \theta.$$

Since this must hold for every x, the operator in the second member must be the zero element of $\mathfrak{E}(\mathfrak{X})$. We have consequently proved *the analogue of Cauchy's theorem for operator functions* and the same argument applies to *vector functions*.

THEOREM 3.10.1. *If $x(\zeta)$ is a holomorphic vector function on the domain D to the (B)-space $\mathfrak{X}$, then*

$$\int_C x(\zeta)\,d\zeta = \theta$$

for every simple closed rectifiable contour C in D such that the interior of C belongs to D. Similarly, if $U(\zeta)$ is a holomorphic operator function on D to $\mathfrak{E}(\mathfrak{X})$ then

$$\int_C U(\zeta)\,d\zeta = \Theta.$$

It is of course immateral if $U(\zeta)$ maps $\mathfrak{X}$ into $\mathfrak{X}$ or into another (B)-space $\mathfrak{Y}$.

The fact that linear operations commute with integration provides us with one of the most powerful tools in extending classical function theory to vector-valued functions. The following very useful theorem is typical for the procedure.

THEOREM 3.10.2. *Let C be a rectifiable curve in the complex plane, let $x(\tau)$ be strongly continuous on C to $\mathfrak{X}$, and let $K(\zeta, \tau)$ be a numerically-valued function with the following properties. There exists a domain D in the ζ-plane such that $K(\zeta, \tau)$ is a holomorphic function of ζ in D for every fixed τ on C and $K(\zeta, \tau)$ is a continuous function of τ on C for every fixed ζ in D. Then*

$$y(\zeta) = \int_C K(\zeta, \tau)x(\tau)\, d\tau$$

is a holomorphic function on D to $\mathfrak{X}$.

REMARK. The assumptions that C is of finite length and that $x(\tau)$ and $K(\zeta, \tau)$ are continuous in τ may be weakened in an obvious manner. Thus in the special but frequently occurring case in which C is an infinite line segment, it is sufficient to assume that $K(\zeta, \tau)$ is a bounded measurable function of τ and that $x(\tau)$ is (B)-integrable over the segment in question.

PROOF. The existence of $y(\zeta)$ as a function on D to $\mathfrak{X}$ is obvious. Since

$$x^*[y(\zeta)] = \int_C K(\zeta, \tau)x^*[x(\tau)]\, d\tau$$

is holomorphic in D in the sense of Cauchy for every $x^* \in \mathfrak{X}^*$, $y(\zeta)$ is holomorphic in D by Definition 3.9.1.

Taking $K(\zeta, \tau) = 1/(\tau - \zeta)$, we obtain integrals of the Cauchy type, which consequently define holomorphic functions of ζ in each of the domains into which C divides the ζ-plane. In particular, this observation gives the Cauchy integral representations of a holomorphic vector function and its derivatives. We have merely to note that if $x(\zeta)$ is holomorphic then

$$\frac{d}{d\zeta}\, x^*[x(\zeta)] = x^*\left[\frac{d}{d\zeta}\, x(\zeta)\right]$$

as is seen from

$$\frac{1}{\alpha}\,\{x^*[x(\zeta + \alpha)] - x^*[x(\zeta)]\} = x^*\left\{\frac{1}{\alpha}\,[x(\zeta + \alpha) - x(\zeta)]\right\}$$

by letting $\alpha \to 0$, remembering that the expression inside the braces on the right tends strongly to $x'(\zeta)$. These observations lead to

THEOREM 3.10.3. *Let $x(\zeta)$ be a holomorphic function on the domain D to the (B)-space $\mathfrak{X}$. Let C be a simple closed rectifiable curve in D, the interior of which is contained in D, such that* $\arg(\tau - \zeta)$ *increases by 2π when τ describes C (positive orientation). Then*

$$x^{(n)}(\zeta) = \frac{n!}{2\pi i}\int_C \frac{x(\tau)\, d\tau}{(\tau - \zeta)^{n+1}}, \qquad n = 0, 1, 2, \cdots.$$

It follows from Theorem 3.10.1 that the integral $\int x(\zeta)d\zeta$ is independent of the path and depends only upon initial and terminal points, if D is simply-connected, and in the multiply-connected case we may deform paths in the

customary manner. Thus we have the usual freedom of choice of paths to suit our needs.

If $x(\zeta)$ is holomorphic in $|\zeta - \zeta_0| < R$ and $\|x(\zeta)\| \leqq M$ for such values of ζ, then taking C to be the circle $|\zeta - \zeta_0| = r < R$ in Theorem 3.10.3 we get the *Cauchy estimates*

$$\|x^{(n)}(\zeta_0)\| \leqq MR^{-n}n!.$$

From these estimates we conclude the validity of the *Taylor expansion*

$$x(\zeta) = \sum_{n=0}^{\infty} \frac{x^{(n)}(\zeta_0)}{n!} (\zeta - \zeta_0)^n \qquad \text{for} \quad |\zeta - \zeta_0| < R.$$

Indeed, the series on the right converges strongly to a limit for such values of ζ and since

$$x^*[x^{(n)}(\zeta)] = \frac{d^n}{d\zeta^n} x^*[x(\zeta)]$$

for every x^*, the series converges weakly to $x(\zeta)$. Its strong limit must then also be $x(\zeta)$. This reasoning exhibits another mode of using the functionals on $\mathfrak{X}$ in order to extend classical function theory to vector functions with values in $\mathfrak{X}$. The same device may be used to prove *Laurent's expansion*:

If $x(\zeta)$ is holomorphic in $0 \leqq R_1 < |\zeta - \zeta_0| < R_2 \leqq \infty$, then

$$x(\zeta) = \sum_{n=-\infty}^{\infty} a_n(\zeta - \zeta_0)^n, \qquad a_n = \frac{1}{2\pi i} \int_C x(\tau)(\tau - \zeta_0)^{-n-1}\, d\tau$$

where C, for instance, is the circle $|\tau - \zeta_0| = R$, $R_1 < R < R_2$.

If $R_1 = 0$ and there is actually at least one $a_n \neq \theta$ with a negative subscript, then $\zeta = \zeta_0$ is a *singular point* of $x(\zeta)$, namely a *pole of order m* if $a_{-m} \neq \theta$ but $a_n = \theta$ for $n < -m$ and otherwise an (isolated) *essential singularity*. In the case of a pole we have

$$M_1 |\zeta - \zeta_0|^{-m} \leqq \|x(\zeta)\| \leqq M_2 |\zeta - \zeta_0|^{-m}$$

for all small values of $|\zeta - \zeta_0|$.

From the Taylor expansion the *uniqueness theorem* is concluded in the usual manner:

THEOREM 3.10.4. *If $x(\zeta)$ and $y(\zeta)$ are holomorphic in D and if $x(\zeta_n) = y(\zeta_n)$, $n = 1, 2, 3, \cdots$, the points $\{\zeta_n\}$ having a limit point in D, then $x(\zeta) \equiv y(\zeta)$.*

The property of being holomorphic is strongly adherent and is preserved by various convergence processes. The next theorem contains the most elementary result in this direction; the theorem of Vitali is given in a later section (Theorem 3.13.2 below).

THEOREM 3.10.5. *Let $\{x_n(\zeta)\}$ be a sequence of holomorphic functions on D to $\mathfrak{X}$ which converges uniformly with respect to ζ on a simple closed rectifiable curve C, the interior of which, D_0 say, is also in D. Then $\{x_n(\zeta)\}$ converges to a holomorphic*

function $x(\zeta)$ *in* D_0 *and, moreover,* $x_n^{(k)}(\zeta) \to x^{(k)}(\zeta)$ *in* D_0 *for every* k. *The convergence is uniform with respect to* ζ, *in* D_0 *when* $k = 0$, *in any fixed domain interior to* D_0 *when* $k > 0$.

The classical proof applies, *mutatis mutandis*, to the vector case.

3.11. Analytic continuation. Theorem 3.10.4 has a number of important consequences, the basic one being that *the Weierstrass principle of analytic continuation applies to vector-valued functions.*

Starting with an element of the function, a power series in $(\zeta - \zeta_0)$ with coefficients in $\mathfrak{X}$ and strongly convergent for $|\zeta - \zeta_0| < \rho(\zeta_0)$, we obtain the totality of *regular elements* by the usual method of continuation. To this set we add the *algebraic elements* corresponding to the algebraic singularities including ordinary poles. *The set of all regular and algebraic elements constitutes the vector-valued analytic function* $x(\zeta)$.

We note that the *theorem of monodromy* is valid for vector-valued analytic functions, that is, *if a regular element of* $x(\zeta)$ *can be continued analytically along every path in a simply-connected domain* D, *then the resulting regular elements form a holomorphic function in* D.

Suppose for the sake of simplicity that the analytic function $x(\zeta)$ is single-valued. Then *the set of points which are centers of regular elements is the domain of holomorphism of* $x(\zeta)$. Adding the set of poles, we get the *domain of existence* of $x(\zeta)$ which in this case coincides with the *domain of meromorphism. Every boundary point of the domain of holomorphism is a singular point of* $x(\zeta)$. The accessible boundary points may be characterized in the usual manner by the fact that the radius of convergence of a regular element tends to zero when the center approaches the boundary point in question. We note that *a singular point of* $x(\zeta)$ *is necessarily a singular point of at least one of the scalar functions* $x^*[x(\zeta)]$. If Σ is the union of the singular points of all functions $x^*[x(\zeta)]$, then the boundary of Σ is the boundary of the domain of holomorphism of $x(\zeta)$, that is, the set of singular points of $x(\zeta)$. A similar situation holds for holomorphic operator functions. A singular point of $U(\zeta)$ is necessarily a singular point of at least one scalar function $x^*\{U(\zeta)[x]\}$ and the boundary of the union of the singular points of all functions $x^*\{U(\zeta)[x]\}$ is the set of singularities of $U(\zeta)$.

The *law of permanency of functional equations* has only limited scope for vector-valued functions since multiplication is not defined in ordinary (B)-spaces. However, it does hold for linear functional equations in the following sense:

Let $T[x; \zeta]$ *be a linear transformation on* $\mathfrak{X}$ *to* $\mathfrak{Y}$, *not necessarily bounded. Suppose there exists a fixed domain* D *in the* ζ*-plane such that if* $x(\zeta)$ *is holomorphic in any domain* $D_0 \subset D$, *then* $T[x(\zeta); \zeta]$ *exists and is holomorphic in* D_0. *Now let* $x(\zeta)$ *be a solution of the equation.*

$$T[x; \zeta] = \theta$$

holomorphic in $D_0 \subset D$. *If* $x(\zeta)$ *can be continued analytically along a path* C *in* D, *then* $T[x(\zeta); \zeta]$ *can be continued along the same path and* $T[x(\zeta); \zeta] \equiv \theta$ *on* C.

The first half of the assertion follows from the initial assumption on T, the second half is then a consequence of Theorem 3.10.4.

3.12. The principle of the maximum. We return to the Cauchy integral representing $x(\zeta)$. If C is taken to be the circle $|\zeta - \zeta_0| = r$, the formula may be written

$$x(\zeta_0) = \frac{1}{2\pi} \int_0^{2\pi} x(\zeta_0 + re^{i\theta})\, d\theta.$$

Consequently

$$||\, x(\zeta_0) \,|| \leqq \frac{1}{2\pi} \int_0^{2\pi} ||\, x(\zeta_0 + re^{i\theta}) \,||\, d\theta.$$

This formula has important implications.

THEOREM 3.12.1. *If $x(\zeta)$ is a holomorphic function on D to $\mathfrak{X}$, then $||\, x(\zeta) \,||$ is a subharmonic function of ζ in D. It follows that $||\, x(\zeta) \,||$ can have no maximum in D and no other minimum than zero.*

This in turn leads to the *principle of the maximum*:

Let $x(\zeta)$ be defined in a domain D and on its boundary C, holomorphic in D and strongly continuous in $D \cup C$. If $\sup ||\, x(\zeta) \,|| = M$ *for ζ on C, then either $x(\zeta)$ is of constant norm M or $||\, x(\zeta) \,|| < M$ in D.*

There is a large group of theorems in classical function theory which is more or less closely attached to the principle of the maximum. The common characteristic of these theorems is that fairly meager information concerning the absolute value of a holomorphic function permits far-reaching conclusions concerning the properties of the function. Such theorems as a rule may be carried over to vector-valued functions, simply replacing statements concerning $|\, f(\zeta) \,|$ by the corresponding statements concerning $||\, x(\zeta) \,||$. Often the classical proofs carry over directly; if not, the assumptions show that the classical theorem applies to all the scalar functions $x^*[x(\zeta)]$ and the desired conclusion for $x(\zeta)$ is reached with the aid of the principle of uniform boundedness or from the fact that the vanishing of all functionals of an element in $\mathfrak{X}$ implies that the element is θ. The following two theorems belong to this group. The first is the *extended theorem of Liouville.* We recall that *a function which is holomorphic in the finite ζ-plane is said to be entire.*

THEOREM 3.12.2. *An entire function $x(\zeta)$ such that $||\, x(re^{i\theta}) \,|| \leqq Mr^{\alpha}$, $0 \leqq \theta \leqq 2\pi$, α fixed $\geqq 0$, for all large r, is a polynomial in ζ of degree $\leqq \alpha$. It is a constant if $\alpha < 1$.*

THEOREM 3.12.3. *If $x(\zeta)$ is holomorphic in $0 < |\, \zeta - \zeta_0 \,| < R$ and if $||\, x(\zeta_0 + re^{i\theta}) \,|| \leqq Mr^{-\alpha}$ for all small values of r, then $\zeta = \zeta_0$ is a pole of order $\leqq \alpha$.*

The results of the classical investigations centering around the *Phragmén-Lindelöf theorem* also admit of extensions to vector-valued functions. A number of these theorems have already proved to be useful in modern functional analysis and will be used in later parts of this treatise. Some of these theorems are listed below without proofs. The reader will find proofs of the classical prototypes and indications of the extensions in the papers quoted at the end of this paragraph. The classical case of Theorem 3.12.4 is due to F. and R. Nevanlinna (extension I. Gelfand); Theorem 3.12.5 is due to F. Carlson in the numerically-valued case (vector extension in the first paper of E. Hille); Theorems 3.12.6

and 3.12.7 go back to G. Pólya (see paper by G. Szegö for proofs, for the vector extension see second paper of E. Hille and M. H. Stone [4]).

THEOREM 3.12.4. *Let $x(\zeta)$ be holomorphic in $\mathfrak{R}(\zeta) > 0$. To every finite point on the imaginary axis and to every $\epsilon > 0$ there shall exist a (semi-circular) neighborhood in which $\|x(\zeta)\| \leqq 1 + \epsilon$. Then either $\|x(\zeta)\| \leqq 1$ for all ζ in $\mathfrak{R}(\zeta) > 0$ or*

$$\liminf_{r\to\infty} \frac{1}{r} \int_{-\pi/2}^{\pi/2} \log^+ \|x(re^{i\theta})\| \cos\theta \, d\theta > 0.$$

Here $\log^+\alpha = \max(\log\alpha, 0)$.

THEOREM 3.12.5. *Let $x(\zeta)$ be holomorphic in $\mathfrak{R}(\zeta) > 0$ and strongly continuous in $\mathfrak{R}(\zeta) \geqq 0$. Suppose that*
(i) $\|x(\pm ir)\| \leqq Ce^{\pi r}$,
(ii) $\|x(re^{i\theta})\| \leqq C\exp[\lambda(\theta)r]$, $-\pi/2 \leqq \theta \leqq \pi/2$, *where* $\lambda(\theta) \leqq M$, $\lambda(-\theta) = \lambda(\theta)$, *and*

$$\limsup_{\delta\to 0} \frac{1}{\delta}\left\{\pi - \lambda\left(\frac{\pi}{2} - \delta\right)\right\} = +\infty, \quad \textit{and}$$

(iii) $x(n) = \theta, n = 1, 2, 3, \cdots$.
Then $x(\zeta) \equiv \theta$.

THEOREM 3.12.6. *If $x(\zeta)$ is an entire function of order one and minimal type and if $\|x(\pm n)\| = o(n^\alpha)$, $\alpha > 0$, then $x(\zeta)$ is a polynomial in ζ of degree $< \alpha$. In particular, $x(\zeta)$ is a constant if $\|x(\pm n)\|$ is bounded with respect to n.*

THEOREM 3.12.7. *If $y(\zeta)$ is an entire function of $1/(\zeta - 1)$ and if in the expansions*

$$y(\zeta) = \sum_0^\infty a_n\zeta^n, \ |\zeta| < 1; \qquad y(\zeta) = \sum_0^\infty b_n\zeta^{-n}, \ |\zeta| > 1,$$

both $\|a_n\|$ and $\|b_n\|$ are $o(n^\beta)$, $\beta > 0$, then $\zeta = 1$ is a pole of order $< \beta + 1$, that is, $y(\zeta)$ is a polynomial in $1/(\zeta - 1)$ of degree $< \beta + 1$.

For vector-valued entire functions $x(\zeta)$ the notions of order and type are defined by obvious modifications of the classical concepts. Thus *$x(\zeta)$ is of order ρ if*

$$\rho = \limsup_{r\to\infty} [\log\log M(r; x)]/[\log r]$$

where $M(r; x) = \max \|x(re^{i\theta})\|$. *A function of order ρ is of type α if*

$$\alpha = \limsup_{r\to\infty} r^{-\rho} \log M(r; x).$$

The function is of minimal type if $\alpha = 0$, normal type if $0 < \alpha < \infty$, maximal type if $\alpha = +\infty$.

3.13. The theorem of Vitali. This name covers two distinct but closely related propositions dealing with functions which are holomorphic and uniformly bounded in a fixed domain. The first asserts that any family of such functions contains a sequence, uniformly convergent in any interior domain; the second that a sequence which converges in a set having a limit point in the domain converges uniformly in any interior domain. These propositions are usually proved on the basis of compactness and the first one seems to require such an argument. The second one does not and the proof of E. Lindelöf [**3**] can be adapted to vector-valued functions. As a preliminary we prove *Schwarz's lemma*:

THEOREM 3.13.1. *Let* $x(\zeta)$ *be holomorphic in* $|\zeta| < 1$, $\|x(\zeta)\| \leqq M$ *and* $x(0) = \theta$. *Then* $\|x(\zeta)\| \leqq M|\zeta|$ *for* $|\zeta| < 1$.

PROOF. The classical form of the lemma shows that $|x^*[x(\zeta)]| \leqq M|\zeta|$ for every functional x^* such that $\|x^*\| = 1$. Theorem 2.9.3 shows that for every fixed ζ we may find an x^* such that $\|x^*\| = 1$ and $x^*[x(\zeta)] = \|x(\zeta)\|$. It follows that also $\|x(\zeta)\| \leqq M|\zeta|$.

THEOREM 3.13.2. *Let* $\{x_n(\zeta)\}$ *be holomorphic in a fixed domain* D *and* $\|x_n(\zeta)\| \leqq M$ *for all* n *and all* ζ *in* D. *Let there be a set* $\{\zeta_k\}$ *in* D *having a limit point* ζ_0 *in* D *such that* $\lim_{n\to\infty} x_n(\zeta_k)$ *exists for every* k. *Then* $\lim_{n\to\infty} x_n(\zeta)$ *exists everywhere in* D, *the convergence is uniform with respect to* ζ *in any fixed bounded domain which has positive distance from the boundary of* D, *and the limit function* $x(\zeta)$ *is holomorphic in* D.

PROOF. We prove the theorem for the following special case. Let D be the interior of the unit circle, $\zeta_0 = 0$, and $M = 1$. We have then

$$x_n(\zeta) = \sum_{m=0}^{\infty} a_{mn}\zeta^m, \qquad |\zeta| < 1,$$

and $\|a_{mn}\| \leqq 1$ for all m and n. If $|\zeta| < \frac{1}{2}$, we have for every n

$$\|x_n(\zeta) - a_{0n}\| \leqq \sum_{m=1}^{\infty} \|a_{mn}\|\,|\zeta|^m < 2|\zeta|.$$

Hence for a fixed but arbitrary k with $|\zeta_k| < \frac{1}{2}$

$$\begin{aligned}\|a_{0n} - a_{0p}\| &\leqq \|x_n(\zeta_k) - a_{0n}\| + \|x_n(\zeta_k) - x_p(\zeta_k)\| + \|x_p(\zeta_k) - a_{0p}\| \\ &\leqq 4|\zeta_k| + \|x_n(\zeta_k) - x_p(\zeta_k)\|\end{aligned}$$

and

$$\limsup_{n,p\to\infty} \|a_{0n} - a_{0p}\| \leqq 4|\zeta_k|.$$

Since $\zeta_k \to 0$ when $k \to \infty$, it follows that $\{a_{0n}\}$ is a Cauchy sequence and has a strong limit, a_0 say, and $\|a_0\| \leqq 1$.

We now put $x_{n,1}(\zeta) = [x_n(\zeta) - a_{0n}]/\zeta$. By Schwarz's lemma $\| x_{n,1}(\zeta) \| \leqq 2$ for all n and all ζ, $|\zeta| < 1$. It follows that the new sequence $\{x_{n,1}(\zeta)\}$ satisfies the same conditions as the original sequence $\{x_n(\zeta)\}$ except that the bound is 2 instead of 1. The bound for the coefficients is obviously unchanged, however. Using the preceding argument we then see that $\lim_{n\to\infty} a_{1n} = a_1$ exists and $\| a_1 \| \leqq 1$. By complete induction we get the existence of $\lim_{n\to\infty} a_{mn} = a_m$ for all m and $\| a_m \| \leqq 1$. Now form

$$x(\zeta) = \sum_{m=0}^{\infty} a_m \zeta^m, \qquad |\zeta| < 1.$$

Then for $|\zeta| \leqq r < 1$

$$\| x_n(\zeta) - x(\zeta) \| \leqq \sum_{m=0}^{k} \| a_{mn} - a_m \| r^m + 2r^{k+1}/(1 - r),$$

whence we conclude that $\| x_n(\zeta) - x(\zeta) \| \to 0$ when $n \to \infty$, uniformly with respect to ζ for $|\zeta| \leqq r < 1$. This completes the proof of the special case.

The extension to the general case is routine analysis. We cover the domain $D_0 \subset D$ by a finite number of circles all in D. The first circle is placed with its center at $\zeta = \zeta_0$, each consecutive circle is placed with its center in a preceding circle. The preceding proof shows the convergence in the first circle, whence it spreads to the consecutive circles and hence to all of D_0. If the centers are properly chosen, the given circles may be replaced by smaller concentric circles, still covering D_0, in which we have uniform convergence.

3.14. Vector-valued functions of several complex variables. A considerable portion of the theory developed in the preceding sections may be extended to the case of functions of several complex variables. In the present section we give only a couple of theorems, due to Max Zorn, which are basic for the discussion in Chapter IV.

Let Z_n be the linear vector space of elements $\zeta = (\zeta_1, \cdots, \zeta_n)$, where $\zeta_1, \cdots, \zeta_n$ are complex numbers and addition and scalar multiplication are defined by the usual conventions

$$\zeta_1 + \zeta_2 = (\zeta_{11} + \zeta_{12}, \cdots, \zeta_{n1} + \zeta_{n2}), \alpha\zeta = (\alpha\zeta_1, \cdots, \alpha\zeta_n).$$

We make Z_n into a (B)-space by defining $\| \zeta \|$ in a convenient manner, for example,

$$\| \zeta \| = \{ |\zeta_1|^2 + \cdots + |\zeta_n|^2 \}^{\frac{1}{2}} \text{ or } \| \zeta \| = \max |\zeta_k|.$$

It is immaterial for the following which definition of the norm is used.

Consider a function $f(\zeta) = f(\zeta_1, \cdots, \zeta_n)$ on Z_n to a (B)-space $\mathfrak{X}$, defined in some domain Δ of Z_n. The partial derivative of $f(\zeta_1, \cdots, \zeta_n)$ with respect to ζ_k is defined by the usual convention

$$\frac{\partial f}{\partial \zeta_k} = \lim_{\alpha_k \to 0} \frac{1}{\alpha_k} [f(\zeta_1, \cdots, \zeta_k + \alpha_k, \cdots, \zeta_n) - f(\zeta_1, \cdots, \zeta_k, \cdots, \zeta_n)],$$

the limit, which is taken in the sense of strong convergence in $\mathfrak{X}$, being independent of the manner in which α_k tends to 0.

THEOREM 3.14.1. *Let $f(\zeta_1, \cdots, \zeta_n)$ on Z_n to $\mathfrak{X}$ have first order partials with respect to each ζ_k when $\zeta = (\zeta_1, \cdots, \zeta_n)$ lies in some domain Δ containing the origin. Then:*

(i) *$f(\zeta_1, \cdots, \zeta_n)$ has partial derivatives of all orders and the mixed partials are independent of the order of differentiation;*

(ii) *$f(\zeta)$ is continuous and bounded on every bounded closed subset of Δ;*

(iii) *if $\| f(\zeta) \| \leqq M$ for $\| \zeta \| \leqq \rho$ and $\sigma < \rho$, then for $\| \zeta \| \leqq \sigma$*

$$\left\| f(\zeta) - f(0) - \sum_{k=1}^{n} \left(\frac{\partial f}{\partial \zeta_k}\right)_0 \zeta_k \right\| \leqq [M/\rho(\rho - \sigma)] \, \| \zeta \|^2 ;$$

(iv) *$f(\zeta, \zeta, \cdots, \zeta)$ is differentiable with respect to ζ.*

PROOF. We may assume that this theorem is known for the special case in which $\mathfrak{X} = Z_1$ and we reduce the general case to this particular instance with the aid of the bounded linear functionals $x^* \in \mathfrak{X}^*$.

(i) The numerical function $x^*[f(\zeta_1, \cdots, \zeta_n)]$ is partially differentiable since strong differentiability implies the weak kind. Hence

$$\frac{\partial}{\partial \zeta_j} x^*[f(\zeta_1, \cdots, \zeta_n)] = x^* \left[\frac{\partial}{\partial \zeta_j} f(\zeta_1, \cdots, \zeta_n)\right]$$

exists and is partially differentiable with respect to ζ_k, $k = 1, 2, \cdots, n$, for every $x^* \in \mathfrak{X}^*$. This implies the existence of second order partials of $f(\zeta_1, \cdots, \zeta_n)$ itself (in the sense of strong differentiability) and proves the first part of assertion (i) since the higher derivatives can be handled by an induction argument. The relation

$$x^* \left\{\frac{\partial^2 f}{\partial \zeta_j \partial \zeta_k}\right\} = \frac{\partial^2}{\partial \zeta_j \partial \zeta_k} x^*[f] = \frac{\partial^2}{\partial \zeta_k \partial \zeta_j} x^*[f] = x^* \left\{\frac{\partial^2 f}{\partial \zeta_k \partial \zeta_j}\right\},$$

which is valid for every $x^* \in \mathfrak{X}^*$, shows that the order of differentiation is immaterial.

(ii) This part as well as (iv) follows from (iii); let us only establish the boundedness as it will be necessary for the proof of (iii).

We consider the numerically-valued functions $x^*[f(\zeta)]$ on an arbitrary closed bounded subset E_0 of Δ. For every fixed x^* the set $| x^*[f(E_0)]|$ is bounded; by the theory of uniform boundedness it follows that $\| f(E_0) \|$ is also bounded.

(iii) Now consider a ρ such that $\| \zeta \| \leqq \rho$ lies in Δ. This set may therefore serve as an E_0 ; let M be the corresponding bound. For the numerical functions $x^*[f(\zeta)]$ this gives

$$| x^*[f(\zeta)] | \leqq M \| x^* \| \text{ in } E_0 .$$

Hence, by the numerical case,

$$\left| x^*[f(\zeta)] - x^*[f(0)] - \sum_{k=1}^{n} \left\{\frac{\partial}{\partial \zeta_k} x^*[f]\right\}_0 \zeta_k \right|$$

$$= \left| x^* \left\{f(\zeta) - f(0) - \sum_{k=1}^{n} \left(\frac{\partial f}{\partial \zeta_k}\right)_0 \zeta_k\right\} \right| \leqq \| x^* \| \, [M/\rho(\rho - \sigma)] \, \| \zeta \|^2$$

for $\| \zeta \| \leqq \sigma < \rho$. An application of Theorem 2.9.3 yields (iii). As was observed above, (iii) implies the continuity of $f(\zeta)$ and part (iv).

It should be observed that we have used only the fact that $x^*[f(\zeta_1, \cdots, \zeta_n)]$ is partially differentiable in Δ. The proof is valid also when $n = 1$ and shows that if $x^*[f(\zeta)]$ is differentiable for $|\zeta| < \rho$ and for all x^*, then $f(\zeta)$ is strongly differentiable.

THEOREM 3.14.2. *Let $\sum_{k=0}^{\infty} P_k(\zeta_1, \zeta_2)$ be a series of homogeneous polynomials*

$$P_k(\zeta_1, \zeta_2) = \sum_{n=0}^{k} a_{kn} \zeta_1^n \zeta_2^{k-n},$$

a_{kn} constants in $\mathfrak{X}$, and let Δ be an open set in the space Z_2. The following facts are then equivalent:

(i) *the series converges at each point of Δ;*

(ii) *the terms P_k are uniformly bounded in any bounded neighborhood Δ_0 of any arbitrary point (ζ_{10}, ζ_{20}) in Δ such that $\bar{\Delta}_0 \subset \Delta$;*

(iii) *every point of Δ has a neighborhood Δ_1 such that $\sum_{k=0}^{\infty} \sup_{\Delta_1} \| P_k(\zeta_1, \zeta_2) \|$ is convergent.*

PROOF. It is easy to show that (ii) implies (iii). For let $\| P_k \| \leqq M$ in Δ_0; then for sufficiently small ϵ, $0 < \epsilon < 1$, the set $(1 - \epsilon)\Delta_0$ will still be a neighborhood of (ζ_{10}, ζ_{20}) and in it we have $\| P_k \| \leqq M(1 - \epsilon)^k$ by virtue of the homogeneity of P_k. That (iii) implies (i) is trivial, but the implication (i) implies (ii) is not. Here we have recourse to the numerical case for which the theorem has been proved by F. Hartogs [1]. He states that the series converges locally uniformly in Δ; this implies uniform convergence on bounded closed subsets.

If $\sum P_k(\zeta_1, \zeta_2)$ is convergent in Δ, then

$$x^* \left\{ \sum_{k=0}^{\infty} P_k(\zeta_1, \zeta_2) \right\} = \sum_{k=0}^{\infty} x^*[P_k(\zeta_1, \zeta_2)]$$

will be a series of numerical polynomials, convergent in Δ and therefore uniformly convergent in any Δ_0 as specified. Applying the theory of uniform boundedness to the set

$$\bigcup_{k=1}^{\infty} \bigcup_{(\zeta_1, \zeta_2) \in \Delta_0} P_k(\zeta_1, \zeta_2),$$

we see that it is bounded, which was to be proved.

References. F. Carlson [1], Dunford [3, 7, 8], Fantappiè [2], Gelfand [4], Hartogs [1], Hilbert [1], Hille [7, 11], Lindelöf [3], Lorch [2, 3], F. and R. Nevanlinna [1], Pólya [3], F. Riesz [2], Stone [4], Szegö [1], A. E. Taylor [5, 7], and N. Wiener [1].

CHAPTER IV

FUNCTIONS ON VECTORS TO VECTORS

4.1. Orientation. In preceding chapters we have studied two different types of abstract functions, viz. functions on vectors to scalars, mainly represented by linear bounded functionals, and functions on scalars to vectors, in particular integrable functions and holomorphic functions. To these types we now add a third one: *functions on vectors to vectors*, and the emphasis will be on functions *analytic* in a sense to be specified later. To simplify matters both the domain and the range will be arbitrary complex Banach spaces though less restrictive assumptions are feasible. In particular, the topology of the domain is immaterial for a considerable portion of the theory.

The theory of analytic functions on vectors to vectors is of recent date, but the Volterra theory of functions of composition can be regarded as a forerunner and most of the basic concepts can be traced back to the founders of functional analysis. The fundamental notions of abstract differentials, polynomials, and power series were introduced by M. Fréchet around 1909. A few years later R. Gâteaux applied the ideas of Fréchet to concrete problems (analytic functionals and functions of infinitely many unknowns) and much of the later development is based upon the brilliant work of Gâteaux which was published posthumously.

The modern theory is the outgrowth of the work of three different schools. A. D. Michal and his pupils, in particular, I. E. Highberg and R. S. Martin, developed a theory of abstract power series, while Angus E. Taylor created a theory of analytic functions based upon the Gâteaux differential. The latter line of approach was followed independently by L. M. Graves and T. H. Hildebrandt who applied the resulting theory to implicit functions and boundary value problems. S. Banach seems to have possessed a theory of analytic operations but of this only some fragments dealing with polynomial operations appear to have been published by Banach himself and by S. Mazur and W. Orlicz. See also G. Suchumlinov for the case of analytic functionals.

In the following we attempt to give a unified presentation of the subject matter, confining ourselves to results which are needed in building up the theory or required in later applications. There are two paragraphs: *Differentiable Functions* and *Analytic Functions*. The reader should also consult sections 5.17 and 22.9 which are concerned with the extension of classical analytic functions from the complex field to a Banach algebra with or without unit element. References are found below; for a more extensive bibliography

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This chapter was written in collaboration with Max Zorn to whom most of the new results are due. Zorn has published his investigations in [1, 2, 3].

the reader is referred to the expository papers of Graves [3], Hyers [1], Michal [1], and Taylor [7].

References. Banach [**3, 4**], Evans [**1**], Fantappiè [**2**], Fréchet [**1, 2, 3, 5**], Gâteaux [**1, 2**], Graves [**1, 2, 3**], Highberg [**1, 2**], Hildebrandt and Graves [**1**], Hyers [**1**], Kerner [**2**], Martin [**1**], Mazur and Orlicz [**1**], Michal [**1**], Michal and Clifford [**1**], Michal and Martin [**1**], Suchumlinov [**1**], Taylor [**1, 3, 4, 6, 7, 8**], and Zorn [**1, 2, 3**].

1. Differentiable Functions

4.2. Multilinear forms and polynomials. In the following we shall be concerned with functions defined on a complex (B)-space $\mathfrak{X}$ and having its values in another such space $\mathfrak{Y}$. In Chapter II we considered linear functions on $\mathfrak{X}$ to $\mathfrak{Y}$. The extension to multilinear functions is immediate.

Definition 4.2.1. *If $x_1, x_2, \cdots, x_n$ are variables in $\mathfrak{X}$, a function $F(x_1, x_2, \cdots, x_n)$ defined for all values of the variables and having values in $\mathfrak{Y}$ is called a symmetric n-linear form, if* (i) *it is linear in each variable separately and* (ii) *it is a symmetric function of the variables. It is said to be continuous if it is continuous in each variable separately.*

Definition 4.2.2. *A function $y = P(x)$ on $\mathfrak{X}$ to $\mathfrak{Y}$ defined for all x is called a polynomial in x of degree m if for all $a, h \in \mathfrak{X}$ and all complex α*

$$P(a + \alpha h) = \sum_{\nu=0}^{m} P_\nu(a, h)\alpha^\nu, \tag{4.2.1}$$

where the $P_\nu(a, h)$ are independent of α. The degree is exactly m if $P_m(a, h) \not\equiv \theta$. $P(x)$ is a power or a homogeneous polynomial of degree n if it is a polynomial and $P(\alpha x) \equiv \alpha^n P(x)$ in x and α.

According to this definition, the zero element is homogeneous of arbitrary degree. This is the only case, however, in which the degree of a power differs from its degree as a polynomial. More precisely expressed: *if $P(x)$ is a polynomial of degree exactly m, if $P(x)$ is homogeneous of degree n, and if $P(x) \not\equiv \theta$, then $m = n$.* The following simple argument, due to Christine S. Williams, starts from the identity $\alpha^m P(a + \alpha^{-1}h) = \alpha^m P[\alpha^{-1}(h + \alpha a)]$. Using the homogeneity on the right and expanding both sides with the aid of (4.2.1), we see that

$$\sum_{\nu=0}^{m} P_\nu(a, h)\alpha^{m-\nu} = \sum_{\nu=0}^{m} P_\nu(h, a)\alpha^{m-n+\nu}.$$

Since neither $P_0(a, h)$ nor $P_m(h, a)$ can vanish identically, this identity requires that $m = n$ and

$$P_\nu(a, h) = P_{n-\nu}(h, a), \qquad \nu = 0, 1, 2, \cdots, n. \tag{4.2.2}$$

Returning to general polynomials and rewriting (4.2.1) as a Newton interpolation polynomial

$$\begin{aligned} P(a + \alpha h) &= \sum_{\nu=0}^{m} \Delta_h^\nu P(a) \binom{\alpha}{\nu}, \\ \Delta_h^\nu P(a) &= \sum_{\mu=0}^{\nu} (-1)^{\nu-\mu} \binom{\nu}{\mu} P(a + \mu h), \end{aligned} \tag{4.2.3}$$

we see that the coefficients $P_\nu(a, h)$ are uniquely determined by $P(x)$. In particular

$$P_0(a, h) = P(a), \qquad P_m(a, h) = \frac{1}{m!} \Delta_h^m P(a). \tag{4.2.4}$$

That the last coefficient is actually independent of a will be shown below.

From (4.2.3) we obtain by complete induction that

$$P(a + \alpha_1 h_1 + \cdots + \alpha_n h_n) = \sum \cdots \sum \Delta_{h_1}^{\nu_1} \cdots \Delta_{h_n}^{\nu_n} P(a) \binom{\alpha_1}{\nu_1} \cdots \binom{\alpha_n}{\nu_n}, \tag{4.2.5}$$

where the increments h_μ are arbitrary elements of $\mathfrak{X}$ and the summations with respect to $\nu_1, \cdots, \nu_n$ go from 0 to m. The right side is a polynomial in the n complex variables $\alpha_1, \cdots, \alpha_n$ of total degree nm. Replacing each α_μ by $\lambda\alpha_\mu$ we obtain a polynomial in λ of degree nm which by formula (4.2.1) must reduce to one of degree m. Owing to the arbitrariness of the α's this is possible if and only if each difference of $P(a)$ involving more than m spans h_μ vanishes identically. This leads to the basic

THEOREM 4.2.1. *If $P(x)$ is a polynomial of degree exactly m, then*

$$\Delta_{h_1 h_2 \cdots h_{m+1}}^{m+1} P(x) = \theta \tag{4.2.6}$$

identically in x and in the increments $h_1, \cdots, h_{m+1}$. No difference of order m can vanish identically. Conversely, if $P(x)$ is a polynomial and if a difference of $P(x)$ of order k vanishes identically in x and in the increments, then the degree of $P(x)$ is less than k.

PROOF. (4.2.6) was proved above. If the degree of $P(x)$ is exactly m, then $m!\, P_m(a, h) = \Delta_h^m P(a)$ does not vanish identically. The converse assertion follows from (4.2.3) which shows that $P(a + \alpha h)$ reduces to a polynomial in α of degree less than k since the identical vanishing of a kth order difference implies the identical vanishing of all differences of order $\geqq k$.

Setting

$$P(x_1, \cdots, x_m) = \frac{1}{m!} \Delta_{x_1 \cdots x_m}^m P(a), \tag{4.2.7}$$

we note that giving a an increment h and taking the difference of span h produces a result which vanishes identically in h. It follows that $P(x_1, \cdots, x_m)$ is actually independent of a. This shows in particular that the coefficient $P_m(a, h)$ is independent of a as asserted above.

It is obvious that any sum of polynomials is a polynomial of a degree not exceeding the highest degree present among the summands. If $P(x)$ is a polynomial in x of exact degree m so is $P(x + c)$ for any fixed c. Further, a polynomial of degree m is the sum of homogeneous polynomials

$$P(x) = \sum_{\nu=0}^{m} P_\nu(x),$$

where $P_\nu(x)$ is homogeneous in x of degree ν. It follows from (4.2.1) that $P_\nu(x) = P_\nu(\theta, x)$ so that $P_\nu(x)$ is uniquely determined.

DEFINITION 4.2.3. *The polar form of a homogeneous polynomial $P(x)$ of degree m is defined by* (4.2.7).

THEOREM 4.2.2. *If $F(x_1, \cdots, x_n)$ is a symmetric n-linear form, then $F(x, \cdots, x)$ is a homogeneous polynomial of degree n.*

PROOF. One shows by induction that

$$F(\lambda a + \mu b, \cdots, \lambda a + \mu b) = \sum_{0}^{n} \binom{n}{\nu} F(a, \cdots, a, b, \cdots, b)\lambda^\nu \mu^{n-\nu}, \tag{4.2.8}$$

where on the right the variables a and b occur ν and $n - \nu$ times respectively. This proves the assertion. A converse is given by

THEOREM 4.2.3. *If $P(x)$ is a homogeneous polynomial of degree n, its polar form is a symmetric n-linear form. In terms of the polar form we have $P(x) = P(x, \cdots, x)$ and*

$$P(\lambda a + \mu b) = \sum_{\nu=0}^{n} \binom{n}{\nu} P(a, \cdots, a, b, \cdots, b)\lambda^\nu \mu^{n-\nu}, \tag{4.2.9}$$

where the coefficient of $\lambda^\nu \mu^{n-\nu}$ is a homogeneous polynomial in a of degree ν and in b of degree $n - \nu$.

PROOF. That the polar form is symmetric follows from (4.2.7). Let us prove that it is linear in x_n. This follows from Theorem 4.2.1 upon observing that

$$f(x) = \frac{1}{n!} \Delta^{n-1}_{x_1 \cdots x_{n-1}} P(x)$$

is a non-null polynomial in x, the second differences of which vanish identically. It follows that $f(x)$ is a polynomial of degree one and hence that $f(x) - f(\theta)$ is homogeneous of degree one. But then $f(x_n) - f(\theta) = P(x_1, \cdots, x_n)$ is linear in x_n. Thus the polar form is n-linear. That $P(x, \cdots, x) = P(x)$ follows from

(4.2.2), (4.2.3) and (4.2.7). Formula (4.2.9) is then an immediate consequence of (4.2.8). The homogeneity properties are obvious.

The rest of this section is devoted to questions of continuity.

THEOREM 4.2.4. *A homogeneous polynomial of degree n is continuous if and only if it is bounded in some sphere. It is then bounded in every fixed finite sphere and satisfies a Lipschitz condition of order one uniformly in such a sphere. Moreover, there exists a constant M such that for all x*

$$\| P(x) \| \leqq M \| x \|^n.$$

PROOF. Suppose that $\| P(x) \| \leqq B$ in $\| x - a \| \leqq \rho$. The coefficients $P_k(a, b)$ in the expression for $P(a + \alpha b)$ given by (4.2.1) can obviously be expressed linearly with numerical coefficients in terms of $P(a + \omega^\nu b)$, $\nu = 0, 1, 2, \cdots, n$, where ω is a primitive $(n + 1)$th root of unity. There is consequently a B_1 such that for $\| b \| \leqq \rho$ we have

$$\| P_k(a, b) \| \leqq B_1, \qquad k = 0, 1, \cdots, n.$$

$P_k(a, b)$ being homogeneous of degree $n - k$ in b, one infers that for all values of b

$$\| P_k(a, b) \| \leqq B_1\rho^{k-n} \| b \|^{n-k}, \qquad k = 0, 1, \cdots, n, \tag{4.2.10}$$

and this shows that $P(x) = P(a + (x - a))$ is bounded in every finite sphere. Since $P_0(a, b) = P(a)$, the estimate also gives

$$\| P(a + h) - P(a) \| = O[\| h \|].$$

Since now (4.2.10) holds uniformly with respect to a in any bounded domain of $\mathfrak{X}$, the Lipschitz condition also holds uniformly in such a domain. For $a = \theta$, $b = x$, $k = 0$ we get the desired inequality $\| P(x) \| \leqq M \| x \|^n$.

Conversely, if $P(x)$ is continuous at $x = a$, then $P(x)$ is bounded in some sphere $\| x - a \| \leqq \rho$ and the preceding argument applies.

THEOREM 4.2.5. *A symmetric n-linear form $F(x_1, \cdots, x_n)$ is continuous if and only if it is bounded in the sense that there is a constant M such that $\| F(x_1, \cdots, x_n) \| \leqq M \| x_1 \| \cdots \| x_n \|$ for all values of the variables. It is then a continuous function of $(x_1, \cdots, x_n)$.*

PROOF. The condition is sufficient since

$$\| F(x_1, \cdots, x_n) - F(a_1, \cdots, a_n) \|$$
$$\leqq \sum_{k=1}^{n} \| F(a_1, \cdots, a_{k-1}, x_k - a_k, x_{k+1}, \cdots, x_n) \| \leqq M\rho^{n-1} \sum_{k=1}^{n} \| x_k - a_k \|$$

if $\| a_k \| \leqq \rho$, $\| x_k \| \leqq \rho$ for all k.

To prove the necessity it is enough to consider the case $n = 2$; the general case can then be handled by complete induction. Thus we have a symmetric bilinear form $F(x, y)$ continuous in x and y separately. It follows that $F(x, \theta) =$

$F(\theta, y) \equiv \theta$. The linearity in y gives $\| F(x, y) \| \leqq M(x) \| y \|$ where the bound $M(x)$ obviously satisfies the conditions

$$M(x_1 + x_2) \leqq M(x_1) + M(x_2),\ M(\alpha x) = | \alpha | M(x),\ M(\theta) = 0.$$

Thus $M(x)$ is a non-negative sublinear functional defined for all x. Let

$$\bar{M}(\theta) = \lim_{\delta \to 0} \sup_{\| x \| \leqq \delta} M(x).$$

We start by proving that $\bar{M}(\theta)$ is finite. If the converse were true, we could find a sequence $\{x_n\}$ such that (i) $x_n \to \theta$, (ii) $M(x_n) > n$. Consider the corresponding sequence of linear functions $F(x_n, y)$. By assumption $F(x_n, y) \to F(\theta, y) = \theta$ for every $y \in \mathfrak{X}$. By the theory of uniform boundedness, this implies that $\| F(x_n, \cdot) \| = M(x_n) \leqq A$ for all n. This contradicts the assumption that $\bar{M}(\theta) = +\infty$. There exists consequently a finite $B = B(\delta)$ such that $M(x) \leqq B(\delta)$ for $\| x \| \leqq \delta$. Hence $M(x) \leqq B(\delta)\delta^{-1} \| x \| \leqq M \| x \|$ for all x and

$$\| F(x, y) \| \leqq M \| x \| \| y \|.$$

Thus $F(x, y)$ is bounded and consequently a continuous function of (x, y) as asserted.

Combining the two preceding theorems one gets:

THEOREM 4.2.6. *A homogeneous polynomial is continuous if and only if its polar form is continuous. Conversely, a symmetric n-linear form $F(x_1, \cdots, x_n)$ is continuous if and only if the polynomial $F(x, \cdots, x)$ is so.*

DEFINITION 4.2.4. *If $P(x)$ is a homogeneous continuous polynomial of degree n, the least value of M for which $\| P(x) \| \leqq M \| x \|^n$ for all x is called the bound or the norm of $P(\cdot)$ and is denoted by $\| P \|$. For a continuous symmetric n-linear form the bound or norm $\| F \|$ is the least value of M such that $\| F(x_1, \cdots, x_n) \| \leqq M \| x_1 \| \cdots \| x_n \|$ for all $x_1, \cdots, x_n$.*

DEFINITION 4.2.5. *A function $y = f(x)$ on $\mathfrak{X}$ to $\mathfrak{Y}$, defined in the set $\mathfrak{D}$, is said to be continuous in the sense of Baire if there exists a set of the first category $\mathfrak{D}_0 \subset \mathfrak{D}$ such that $f(x)$ is continuous in the set $\mathfrak{D} - \mathfrak{D}_0$.*

If $\mathfrak{Y}$ is separable, this notion is equivalent to $f(x)$ having the property of Baire, that is, for every open set $\mathfrak{G}$ the set $f^{-1}(\mathfrak{G})$ has the property of Baire in the sense defined in section 1.2. For these different notions, we refer to C. Kuratowski [**2**, p. 191 et seq.]. Our use of the notion of continuity in the sense of Baire will be restricted to the implications of the next three theorems; for the first of these we refer to Kuratowski (loc. cit.), for the others to S. Mazur and W. Orlicz [**1**, p. 182].

THEOREM 4.2.7. *A convergent sequence of functions, continuous in the sense of Baire, converges to a function having the same property.*

THEOREM 4.2.8. *If a homogeneous polynomial is continuous in the sense of Baire, then it is continuous everywhere. A symmetric n-linear form which is continuous in the sense of Baire with respect to each variable separately is continuous everywhere with respect to the variables jointly.*

Combining these results we obtain:

THEOREM 4.2.9. *A convergent sequence of continuous symmetric n-linear forms converges to a form with the same properties or the limit vanishes identically. Also, a convergent sequence of continuous homogeneous polynomials of fixed degree n converges either to such a polynomial or to the zero element.*

4.3. G-Differentiability. The crux of the theory of vector-valued functions of vectors is the question of differentiability. In this general situation the differentials of Fréchet and of Gâteaux appear to be the most appropriate concepts. We start by formulating some topological notions of importance to the theory.

DEFINITION 4.3.1. *A subset $\mathfrak{D}$ of a (B)-space $\mathfrak{X}$ is said to be finitely open if for each choice of elements $x_0 \in \mathfrak{D}$ and $h_1, \cdots, h_n \in \mathfrak{X}$, the elements $x_0 + \sum_1^n \zeta_k h_k$ which are in $\mathfrak{D}$ correspond to an open subset of the space Z_n of ordered n-tuples of complex numbers $(\zeta_1, \cdots, \zeta_n)$.*

This is equivalent to saying that $\mathfrak{D}$ *intersects every finite-dimensional linear subspace of $\mathfrak{X}$ in a relatively open set.*

There are actually sets which are finitely open without being open. The following example is due to E. G. Begle and H. Pollard. Let $\{x_n\}$ be a sequence of points in $\mathfrak{X}$ such that (i) for each n, the point x_{n+1} is not in the closed linear vector space spanned by x_1, $x_2, \cdots, x_n$, and (ii) $\lim x_n = x_0$ exists. Let $\mathfrak{D}$ be the complement of the set $\{x_n\}$. Since $\mathfrak{D}$ contains x_0, it cannot be open. On the other hand, any k-dimensional subspace of $\mathfrak{X}$ can contain at most k points x_n; its intersection with $\mathfrak{D}$ is consequently relatively open, that is, $\mathfrak{D}$ is finitely open.

DEFINITION 4.3.2. *A set $\mathfrak{S}^*(x_0)$ is called a c-star about x_0 if $\mathfrak{S}^*(x_0) = x_0 + H$, where $h \in H$, $|\zeta| \leqq 1$ implies that $\zeta h \in H$.*

We note that if $\mathfrak{D}$ is finitely open and contains x_0, then it contains a finitely open c-star about x_0 formed by all points $x_0 + h$ for which $|\zeta| \leqq 1$ implies $x_0 + \zeta h \in \mathfrak{D}$. We speak of the c-star in $\mathfrak{D}$ about x_0.

Let us denote by $\rho(x, h)$ the supremum of all numbers ρ such that $|\zeta| \leqq \rho$ implies that $x + \zeta h \in \mathfrak{D}$.

DEFINITION 4.3.3. *Let $y = f(x)$ on $\mathfrak{X}$ to $\mathfrak{Y}$ be defined in the finitely open set $\mathfrak{D}$ and suppose that for every $x \in \mathfrak{D}$ and $h \in \mathfrak{X}$ the quotient $[f(x + \zeta h) - f(x)]/\zeta$, which is defined for $0 < |\zeta| < \rho(x, h)$, tends to a unique limit as $\zeta \to 0$. We then say that*

(i) *$f(x)$ is G-differentiable in $\mathfrak{D}$;*

(ii)
$$\delta f(x; h) = \delta_x^h f = \lim_{\zeta \to 0} \frac{1}{\zeta} [f(x + \zeta h) - f(x)]$$

is the first variation with increment h of $f(x)$;

(iii) *$f(x)$ possesses a Gâteaux differential in $\mathfrak{D}$.*

Definition 4.3.4. *We say that $f(x)$ is F-differentiable and possesses a total or a Fréchet differential in $\mathfrak{D}$ if* (i) *$\mathfrak{D}$ is open,* (ii) *$\delta f(x; h)$ exists as a continuous function of h, and*

(iii)
$$\lim_{||h|| \to 0} \frac{1}{||h||} || f(x + h) - f(x) - \delta f(x, h) || = 0, \qquad x \in \mathfrak{D}.$$

We note that if f and g are G-differentiable so is $\alpha f + \beta g$ and

$$\delta_x^h[\alpha f + \beta g] = \alpha \delta_x^h f + \beta \delta_x^h g.$$

It is also a simple matter to show that

$$\delta f(x; \alpha h) = \alpha \delta f(x; h)$$

so that $\delta f(x, h)$ *is homogeneous of degree one in* h.

Theorem 4.3.1. *$f(x)$ is G-differentiable in the finitely open set $\mathfrak{D}$ if and only if for every $x \in \mathfrak{D}$ and $h \in \mathfrak{X}$, $f(x + \zeta h)$ is a holomorphic function of ζ when $|\zeta| < \rho(x, h)$.*

Proof. This follows from the observation

$$\left[\frac{d}{d\zeta} f(x + (\zeta_0 + \zeta)h)\right]_{\zeta=0} = \lim_{\zeta \to 0} \frac{1}{\zeta} \{f[(x + \zeta_0 h) + \zeta h] - f(x + \zeta_0 h)\}$$
$$= \delta f(x + \zeta_0 h; h).$$

Corollary. *$f(x)$ is G-differentiable in $\mathfrak{D}$ if and only if for every $x \in \mathfrak{D}$ and any $h_1, \cdots, h_n \in \mathfrak{X}$ the function $f(x + \sum_1^n \zeta_k h_k)$ is partially differentiable with respect to ζ_k, $k = 1, \cdots, n$, in the open set $\Delta = \Delta(h_1, \cdots, h_n)$ of the space Z_n which corresponds to points $x + \sum_1^n \zeta_k h_k$ in $\mathfrak{D}$.*

Theorem 4.3.2. *For every $h \in \mathfrak{X}$, $\delta_x^h f = \delta f(x; h)$ is a G-differentiable function of x.*

Proof. We have

$$\delta_x^{h_2} \delta_x^{h_1} f(x) = \delta_x^{h_2} \left\{\frac{d}{d\zeta} f(x + \zeta h_1)\right\}_{\zeta=0}$$
$$= \left\{\frac{\partial}{\partial \zeta_2} \left[\frac{\partial}{\partial \zeta_1} f(x + \zeta_1 h_1 + \zeta_2 h_2)\right]_{\zeta_1=0}\right\}_{\zeta_2=0}$$
$$= \left\{\frac{\partial^2}{\partial \zeta_1 \partial \zeta_2} f(x + \zeta_1 h_1 + \zeta_2 h_2)\right\}_{0,0}.$$

By the preceding theorem $f(x + \zeta_1 h_1 + \zeta_2 h_2)$ is a partially differentiable function and Theorem 3.14.1 ensures the existence of the higher partials.

We may thus define the nth variation $\delta^n f(x; h_1, \cdots, h_n)$ of $f(x)$ with increments $h_1, \cdots, h_n$ by

DEFINITION 4.3.5. *We set* $\delta^1 f(x; h_1) = \delta f(x; h_1)$; $\delta^{n+1} f(x; h_1, \cdots, h_{n+1}) = \delta_x^{h_{n+1}}[\delta^n f(x; h_1, \cdots, h_n)]$; $\delta^n f(x; h) = \delta^n f(x; h, \cdots, h)$.

For the sake of convenience we add the convention that $[\delta^n f(x; h_1, \cdots, h_n)]_{n=0} = f(x)$. We state without proof

THEOREM 4.3.3. *We have*

$$\text{(i)} \qquad \left\{\frac{\partial}{\partial \zeta_1} f(x + \zeta_1 h_1 + \zeta_2 h_2)\right\}_{0,0} = \delta f(x; h_1);$$

$$\text{(ii)} \qquad \left\{\frac{\partial^n}{\partial \zeta_1 \cdots \partial \zeta_n} f\left(x + \sum_{k=1}^{n} \zeta_k h_k\right)\right\}_{0,\cdots,0} = \delta^n f(x; h_1, \cdots, h_n);$$

$$\text{(iii)} \qquad \left\{\frac{\partial^n}{\partial \zeta_1 \cdots \partial \zeta_n} f\left(x + \sum_{k=1}^{n} \zeta_k h_k\right)\right\}_{h,\cdots,h,0,\cdots,0} = \left\{\frac{d^n}{d\zeta^n} f(x + \zeta h)\right\}_{\zeta=0}.$$

THEOREM 4.3.4. $\delta_x^h f = \delta f(x; h)$ *is linear in* h.

PROOF. We have already noted that $\delta f(x; h)$ is homogeneous of degree one in h, but it remains to show the additivity. Consider the function $f(x + \zeta_1 h_1 + \zeta_2 h_2)$ of $\zeta = (\zeta_1, \zeta_2)$. It is partially differentiable, thus by Theorem 3.14.1 (iii)

$$f(x + \zeta_1 h_1 + \zeta_2 h_2) = f(x) + \zeta_1 \left(\frac{\partial f}{\partial \zeta_1}\right)_0 + \zeta_2 \left(\frac{\partial f}{\partial \zeta_2}\right)_0 + o(\| \zeta \|).$$

By Theorem 4.3.3 (i) this gives

$$f(x + \zeta_1 h_1 + \zeta_2 h_2) = f(x) + \zeta_1 \delta f(x; h_1) + \zeta_2 \delta f(x; h_2) + o(\| \zeta \|).$$

Letting $\zeta_1 = \zeta_2 = \zeta$ we get

$$f[x + \zeta(h_1 + h_2)] - f(x) = \zeta[\delta f(x; h_1) + \delta f(x; h_2)] + o(| \zeta |)$$

since $o(\| \zeta \|) = o(| \zeta |)$. This shows that

$$\begin{aligned} \delta f(x; h_1 + h_2) &= \lim_{\zeta \to 0} \frac{1}{\zeta} [f(x + \zeta(h_1 + h_2)) - f(x)] \\ &= \delta f(x; h_1) + \delta f(x; h_2) \end{aligned}$$

and the theorem is proved.

In defining F-differentiability it is customary to require that $\delta f(x; h)$ exist as a *linear* continuous function of h. Theorem 4.3.4 shows that the assumption of linearity is redundant and this is why it was omitted in Definition 4.3.4. Actually the latter still contains a redundance: Zorn [3] has recently shown that *condition* (iii) *is implied by* (i) *and* (ii). If (ii) is satisfied, we see that $\delta f(x; \cdot)$ *for fixed* x *in* $\mathfrak{D}$ *is a bounded linear operator on* $\mathfrak{X}$ *to* $\mathfrak{Y}$. If the (B)-space of all

such operators be denoted by $\mathfrak{E}[\mathfrak{X}, \mathfrak{Y}]$, then $\delta f(x; \cdot)$ *is a function on* $\mathfrak{D}$ *to* $\mathfrak{E}[\mathfrak{X}, \mathfrak{Y}]$ which, as we shall see later, is continuous and even analytic in x. Zorn calls this function the *derivative* of $f(x)$ and denotes it by $f'(x)$ so that $\delta f(x; h) = f'(x)[h]$.

The notion of a derivative as distinguished from that of a differential appears to have been considered by A. D. Michal in 1936.

THEOREM 4.3.5. *$\delta^n f(x; h_1, \cdots, h_n)$ is a symmetric n-linear form in $h_1, \cdots, h_n$ which is G-differentiable with respect to x when $x \in \mathfrak{D}$. $\delta^n f(x; h)$ is a homogeneous polynomial of degree n in h.*

PROOF. $\delta^n f(x; h_1, \cdots, h_n)$ is linear in the last argument since it is a variation; it is symmetric by Theorem 4.3.3 (ii) and hence linear in each h_k. The rest follows from Theorems 4.2.1 and 4.3.2.

Let us now look at the Taylor development of the function $f(x + \zeta h)$ which must be valid for $|\zeta| < \rho(x, h)$:

$$f(x + \zeta h) = \sum_{n=0}^{\infty} \left\{ \frac{d^n}{d\alpha^n} f(x + \alpha h) \right\}_{\alpha=0} \frac{\zeta^n}{n!}.$$

Using Theorem 4.3.3 (iii) we get

$$f(x + \zeta h) = \sum_{n=0}^{\infty} \delta^n f(x; h) \frac{\zeta^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n f(x; \zeta h).$$

Here we may replace ζh by h; reinterpreting the condition on h geometrically we arrive at

THEOREM 4.3.6. *For $x \in \mathfrak{D}$ and $x + h$ in the c-star about x in $\mathfrak{D}$, we have*

$$f(x + h) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n f(x; h).$$

Another way of stating the condition is to say that *the series converges for* $\rho(x, h) < 1$.

From Theorem 3.10.3 it follows that

$$\delta^n f(x; h) = \frac{n!}{2\pi i} \int_{\Gamma} f(x + \zeta h) \zeta^{-n-1} \, d\zeta, \tag{4.3.1}$$

where Γ is any circle $|\zeta| = \rho'$ with $\rho' < \rho(x, h)$. In particular, if $x + h$ belongs to a c-star in $\mathfrak{D}$ about x, we may choose $\rho' = 1$. This leads to the analog of the classical estimates of Cauchy:

THEOREM 4.3.7. *In any c-star $\mathfrak{C}^*(a)$ about $x = a$ in which $\| f(x) \| \leq M$, we have $\| \delta^n f(a; h) \| \leq Mn!$ for $a + h \in \mathfrak{C}^*(a)$.*

The existence of a finitely open c-star of boundedness may be asserted for every point $a \in \mathfrak{D}$. In the following theorem we have taken $a = \theta$.

THEOREM 4.3.8. *Let* $\mathfrak{C}^*(\theta) \subset \mathfrak{D}$ *be a finitely open c-star about* θ, $\mathfrak{S}^*$ *a subset of* $\mathfrak{C}^*(\theta)$ *which is contained in a finite-dimensional subspace of* $\mathfrak{X}$ *and compact there. Then there exist quantities* ϵ *and* M, $0 < \epsilon < 1$, $0 < M$, *and a finitely open c-star* $\mathfrak{C}_0^*(\theta) = \mathfrak{C}$ *such that* (i) $\mathfrak{S} \subset (1 - \epsilon)\mathfrak{C} \equiv \mathfrak{C}(\epsilon) \subset \mathfrak{C} \subset \mathfrak{C}^*(\theta)$, (ii) $\|f(x)\| \leqq M$ *in* $\mathfrak{C}$, *and* (iii)

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sup_{x \in \mathfrak{C}(\epsilon)} \|\delta^n f(\theta; x)\| < \infty.$$

PROOF. The function $\varphi(x) = \max_{|\zeta| \leq 1} \|f(\zeta x)\|$ is well defined when $x \in \mathfrak{C}^*(\theta)$. If $\mathfrak{X}_n$ is any finite-dimensional linear subspace of $\mathfrak{X}$, then $f(x)$ is continuous in $\mathfrak{X}_n \cap \mathfrak{C}^*(\theta) \equiv \mathfrak{C}_n$ since every x in $\mathfrak{C}_n$ is of the form $\sum_1^n \zeta_k h_k$ and $f(\sum_1^n \zeta_k h_k)$ is a partially differentiable function of $\zeta_1, \cdots, \zeta_n$ for $\sum_1^n \zeta_k h_k$ in $\mathfrak{C}_n$. This implies, however, that $\varphi(x)$ is also continuous in $\mathfrak{C}_n$. In order to see this, let $\{x_k\} \in \mathfrak{C}_n$ and $x_k \to x_0 \in \mathfrak{C}_n$ when $k \to \infty$. There exist complex numbers ζ_k and ζ_0 of absolute value one such that $\varphi(x_k) = \|f(\zeta_k x_k)\|$, $\varphi(x_0) = \|f(\zeta_0 x_0)\|$. Then

$$\limsup \varphi(x_k) = \limsup \|f(\zeta_k x_k)\| \leqq \max_{|\zeta| \leq 1} \|f(\zeta x_0)\| = \varphi(x_0).$$

On the other hand, $\varphi(x_k) \geqq \|f(\zeta_0 x_k)\|$ whence

$$\liminf \varphi(x_k) \geqq \|f(\zeta_0 x_0)\| = \varphi(x_0)$$

and, finally, $\lim \varphi(x_k) = \varphi(x_0)$.

This being established, we consider the subset $\mathfrak{D}(x_0)$ of $\mathfrak{C}^*(\theta)$ where $\varphi(x) < \varphi(x_0) + 1$, $x_0 \in \mathfrak{S}$. This set is finitely open since its intersection with any $\mathfrak{X}_n$ is relatively open; it is a c-star from the definition of $\varphi(x)$ and it contains x_0. $\mathfrak{S}$ being compact, there exists a finite subset of $\mathfrak{S}$ such that the corresponding sets $\mathfrak{D}(x_1), \cdots, \mathfrak{D}(x_n)$ suffice to cover $\mathfrak{S}$. We put $\mathfrak{C} = \bigcup \mathfrak{D}(x_k)$, $M = 1 + \max_k \varphi(x_k)$. Then $\mathfrak{C}$ is a finitely open c-star in $\mathfrak{C}^*(\theta)$, containing $\mathfrak{S}$, and $\|f(x)\| \leqq M$ in $\mathfrak{C}$. Let $\mathfrak{S}$ be contained in the n-dimensional linear subspace $\mathfrak{X}_n$; since $\mathfrak{C} \cap \mathfrak{X}_n$ is relatively open, we can find an ϵ, $0 < \epsilon < 1$, such that $\mathfrak{S} \subset (1 - \epsilon)\mathfrak{C} \equiv \mathfrak{C}(\epsilon)$. For $x \in \mathfrak{C}(\epsilon)$ we can use formula (4.3.1) with x replaced by θ, h by x, and ρ' by $1/(1 - \epsilon)$. It follows that $\sup_{x \in \mathfrak{C}(\epsilon)} \|\delta^n f(\theta; x)\| \leqq M(1 - \epsilon)^n n!$. This completes the proof.

At this juncture an examination of the properties of differentiability of powers is in order.

THEOREM 4.3.9. *A homogeneous polynomial is always G-differentiable; it is F-differentiable if and only if it is continuous.*

PROOF. The first assertion is an immediate consequence of Definition 4.2.2. Formula (4.2.1) shows that if $P(x)$ is of degree n then

$$\delta^k P(x; h) = k! \, P_{n-k}(x, h), \qquad k \leqq n,$$

and that the variations of order greater than n vanish identically. Definition 4.3.4 shows that the continuity of $P(x)$ is necessary for F-differentiability. But if $P(x)$ is continuous, then the homogeneous polynomials $P_{n-k}(x, h)$ are continuous functions of x as well as of h since they may be expressed linearly with constant coefficients in terms of $(n + 1)$ suitably chosen functions $P(x + \alpha_j h)$. Since $P_{n-k}(x, h)$ is homogeneous of degree k in h and of degree $n - k$ in x, we can find a constant M such that

$$\| P_{n-k}(x, h) \| \leqq M \| x \|^{n-k} \| h \|^k, \qquad k = 0, 1, \cdots, n.$$

These estimates show that the conditions of Definition 4.3.4 are satisfied so that $P(x)$ is F-differentiable when it is continuous.

From this point of view *we may refer to a homogeneous polynomial as an F-power or as a G-power according as it is F-differentiable or merely G-differentiable*, that is, according as it is continuous or not. By Theorem 4.2.3 we may replace continuity by boundedness in this statement.

The Taylor series of $f(x)$ given in Theorem 4.3.6 is an expansion in terms of G-powers in h. It could consequently be described as a *G-power series*. Similarly a convergent series in terms of F-powers could be called an *F-power series*. It is suggestive for purposes of comparison to reformulate the main results of this section in the new terminology.

THEOREM 4.3.10. *If $f(x)$ is defined and G-differentiable in a finitely open set $\mathfrak{D} \subset \mathfrak{X}$, then to every $x \in \mathfrak{D}$ there is a G-power series $\sum_{n=0}^{\infty} P_n(x, h)$ where $P_n(x, h)$ is a G-power in h of degree n and a G-differentiable function of x. The functions $P_n(x, h)$ are uniquely determined by $f(x)$ and $P_n(x, h) = \delta^n f(x; h)/n!$. We have*

$$f(x + h) = \sum_{n=0}^{\infty} P_n(x, h),$$

valid if h is in the c-star in $\mathfrak{D}$ about x. To every x there is a finitely open c-star in which $f(x)$ is bounded and the series converges uniformly and normally in the sense that $\sum_{n=0}^{\infty} \sup_{x+h \in \mathfrak{S}} \| P_n(x, h) \|$ converges.

The question of when a G-power series becomes an F-power series will be considered later.

4.4. G-power series. Let $\{P_n(x)\}$, $n = 0, 1, 2, \cdots$, be a given sequence of G-powers in x, the degree of $P_n(x)$ being n. We shall study the G-power series

$$\sum_{n=0}^{\infty} P_n(x), \tag{4.4.1}$$

its region of convergence and the properties of its sum.

We start from a local point of view which enables us to determine the cross sections of the *region of convergence*, denoted by $\mathfrak{C}[P_n]$, with planes through its center $x = \theta$.

Definition 4.4.1. *Let $u \in \mathfrak{X}$ and $\| u \| = 1$ and put*

$$\mu(u) = \limsup_{n\to\infty} \| P_n(u) \|^{1/n}, \qquad \rho(u) = 1/\mu(u).$$

We state without proof

Theorem 4.4.1. (i) *If the terms of* (4.4.1) *are bounded for $x = x_0$, then the series is absolutely convergent for $x = \zeta x_0$, $|\zeta| < 1$.* (ii) *If $x = \zeta u$, $\| u \| = 1$, then the series converges absolutely for $|\zeta| < \rho(u)$ and diverges for $|\zeta| > \rho(u)$.*

Corollary. *The closure of the region of convergence of a $\dot{G}$-power series is a c-star about θ (or reduces to θ).*

The properties of $\rho(u)$ appear to be very complicated even in cases which look fairly simple on the surface. As an example we may take $\mathfrak{X} = (l_2)$, the space of sequences $x = \{\alpha_n\}$, with $\| x \| = \{\sum |\alpha_n|^2\}^{1/2}$, and $\mathfrak{Y} = Z_1$, the space of complex numbers with the usual norm. Then

$$f(x) = \sum_{n=1}^{\infty} (n\alpha_n)^n, \qquad x = \{\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots\}$$

is an F-power series as is easily verified. Here the function $\rho(u)$ has the property that its lower and upper limit functions are identically 0 and $+\infty$ respectively. Indeed, suppose that $u = \{\alpha_n\}$ and $\| u \| = 1$. To any given $\epsilon > 0$ we may find an integer N and two numbers A and B near unity such that

$$\sum_{N+1}^{\infty} |\alpha_n|^2 < \epsilon^2$$

and

$$u_1 = \{A\alpha_1, A\alpha_2, \cdots, A\alpha_N, 0, 0, \cdots\},$$

$$u_2 = B\{\alpha_1, \alpha_2, \cdots, \alpha_N, (N+1)^{-2/3}, (N+2)^{-2/3}, \cdots\}$$

are unit vectors. Here $\rho(u_1) = \infty$ and $\rho(u_2) = 0$, further

$$\| u - u_1 \| \leqq A - 1 + \epsilon, \| u - u_2 \| \leqq 2BN^{-1/6} + |B - 1| + \epsilon,$$

so that u_1 and u_2 are as close to u as we please. This proves the assertion.

A satisfactory theory is obtainable in the case in which there exists a finitely open domain of convergence. This assumption implies certain restrictions on $\rho(u)$ (essentially lower semi-continuity) which, however, we shall not investigate here. We start with a simple lemma.

Lemma 4.4.1. *If a G-power $P(x)$ satisfies $\| P(x) \| \leqq M$ in a c-star $\mathfrak{S}^*(x_0) = x_0 + H$, then it satisfies the same inequality in the c-star $\mathfrak{S}^*(\theta) = \zeta x_0 + H$ for $|\zeta| \leqq 1$.*

Proof. Let $h_0 \in H$ and consider the function $P(\zeta x_0 + h_0)$ for $|\zeta| \leqq 1$. It is holomorphic for all such values of ζ since it is a polynomial in ζ. By the principle of the maximum $\| P(\zeta x_0 + h_0) \|$ reaches its largest value on the unit circle, say for $\zeta = \zeta_0$. But the homogeneity of $P(x)$ gives

$$\| P(\zeta_0 x_0 + h_0) \| = \| P(x_0 + \zeta_0^{-1} h_0) \| \leqq \max_{h\in H} \| P(x_0 + h) \| \leqq M$$

since H is a c-star. This proves the lemma.

THEOREM 4.4.2. *If the G-power series (4.4.1) converges in the finitely open c-star* $\mathfrak{S}^*(x_0) = x_0 + H$, *it will also converge in* $\mathfrak{S}^*(\theta) = \zeta x_0 + H$, $|\zeta| \leqq 1$.

PROOF. Let $h \in H$ and consider the linear subspace $\{\zeta_1 x_0 + \zeta_2 h\} = \{x_0 + h + (\zeta_1 - 1)x_0 + (\zeta_2 - 1)h\}$. It intersects $\mathfrak{S}^*(x_0)$ in a relatively open set $\mathfrak{D}_0$ corresponding to an open set Δ in Z_2. Since $(1, 1) \in \Delta$, we can find an $\epsilon > 0$ such that the closed bicylinder $\Gamma_2 : |\zeta_1 - 1| \leqq \epsilon$, $|\zeta_2 - 1| \leqq \epsilon$ is in Δ. Substituting $x = \zeta_1 x_0 + \zeta_2 h$ into the power series, one obtains a series of homogeneous polynomials in (ζ_1, ζ_2) convergent in Δ. By Theorem 3.14.2 (ii) this implies that the terms of the series are uniformly bounded in Γ_2. Thus there is a finite M such that $\| P_k(x) \| \leqq M$ for x in the c-star $\mathfrak{S}^*(x_0 + h)$: $x_0 + h + \eta_1 x + \eta_2 h$ with $(\eta_1 + 1, \eta_2 + 1) \in \Gamma_2$. By Lemma 4.4.1 the inequality $\| P_k(x) \| \leqq M$ holds also in the c-star $\mathfrak{S}^*(\theta)$: $\zeta(x_0 + h) + \eta_1 x_0 + \eta_2 h$, $|\zeta| \leqq 1$, $(\eta_1 + 1, \eta_2 + 1) \in \Gamma_2$. Using the homogeneity of $P_k(x)$ or referring to Theorem 3.14.2 again, we see that the power series is absolutely convergent in the c-star $\rho\mathfrak{S}^*(\theta)$ for any $\rho < 1$. This c-star, for ρ sufficiently near to 1, contains all points of the form $\zeta_1 x_0 + \zeta_2 h$, $|\zeta_1| \leqq 1$, $|\zeta_2| \leqq 1$. Since $h \in H$ is arbitrary, it follows that the power series converges in the finitely open c-star $\zeta x_0 + H$, $|\zeta| \leqq 1$, as asserted.

COROLLARY. *If a G-power series converges in an open set, it converges in a neighborhood of* θ.

THEOREM 4.4.3. *If the G-power series (4.4.1) converges in a finitely open set* $\mathfrak{D}$, *it will converge uniformly on discs, that is, for fixed* $x \in \mathfrak{D}$, $h \in \mathfrak{X}$ *the series*

$$\sum_0^\infty P_k(x + \zeta h)$$

will be uniformly convergent for $|\zeta| \leqq \rho' < \rho(x, h)$.

PROOF. Consider the series $\sum_0^\infty P_k(\zeta_1 x + \zeta_2 h)$; it is a series of homogeneous polynomials in (ζ_1, ζ_2) which converges in an open set Δ in Z_2, containing all points $(1, \zeta_2)$ with $|\zeta_2| < \rho(x, h)$. The subset defined by $|\zeta_2| \leqq \rho' < \rho(x, h)$ will be a region of uniform convergence by Theorem 3.14.2 (iii).

THEOREM 4.4.4. *If a sequence of functions* $\{f_n(x)\}$ *converges uniformly on the discs* $x + \zeta h$, $|\zeta| \leqq \rho' < \rho(x, h)$, *of a finitely open set* $\mathfrak{D}$ *and if the functions* $f_n(x)$ *are G-differentiable in* $\mathfrak{D}$, *then the limit will also be G-differentiable in* $\mathfrak{D}$.

PROOF. If $f(x) = \lim_{n\to\infty} f_n(x)$, we have to show that $f(x_0 + \zeta h)$ is differentiable with respect to ζ for $|\zeta| < \rho(x_0, h)$. This follows from Theorem 3.10.5. The same theorem also yields:

THEOREM 4.4.5. *Under the conditions of the preceding theorem we have for* $x \in \mathfrak{D}$, $h \in \mathfrak{X}$

$$\delta^k f(x; h) = \lim_{n\to\infty} \delta^k f_n(x; h).$$

Combining Theorems 4.4.3 and 4.4.4 we obtain:

THEOREM 4.4.6. *If a G-power series converges in a finitely open set, its sum will be a G-differentiable function.*

In the preceding theorem we may replace strong convergence by weak.

THEOREM 4.4.7. *If a G-power series converges weakly to a limit in a finitely open set, then its weak sum is G-differentiable.*

PROOF. By assumption the series $\sum_0^\infty y^*[P_n(x)]$ converges in $\mathfrak{D}$ for every $y^* \in \mathfrak{Y}^*$. We note that $y^*[P_n(x)]$ is a G-power in x with values in Z_1 and the series $\sum_0^\infty y^*[P_n(x)]$ converges in the sense of the strong metric of Z_1 (that is, in the sense of Cauchy). By Theorem 4.4.6, $\sum_0^\infty y^*[P_n(x)]$ is G-differentiable in $\mathfrak{D}$ and if $f(x)$ is the weak sum of the series $\sum_0^\infty P_n(x)$, we have consequently that $y^*[f(x)]$ is G-differentiable for every y^*. This is equivalent to $y^*[f(x + \zeta h)]$ being a holomorphic function of ζ in $|\zeta| < \rho(x, h)$ for every y^* and this implies that $f(x + \zeta h)$ is holomorphic in the same domain. It follows that $f(x)$ is G-differentiable in $\mathfrak{D}$ (in the strong sense). We may however go still further:

THEOREM 4.4.8. *If a G-power series converges weakly to a limit in a finitely open set $\mathfrak{D}$, then it converges strongly in $\mathfrak{D}$.*

PROOF. We shall prove this under the additional assumption that $\mathfrak{D}$ is a c-star about θ. It will be seen below that this is no real restriction of the generality. Let the weak sum of the G-power series be $f(x)$. It is G-differentiable in $\mathfrak{D}$ and may consequently be expanded in a Maclaurin series, say $f(x) = \sum_0^\infty P_k(\theta, x)$. This series converges in $\mathfrak{D}$ by Theorem 4.3.10. It follows that for every $y^* \in \mathfrak{Y}^*$

$$\sum_0^\infty y^*[P_k(\theta, x)] = \sum_0^\infty y^*[P_k(x)].$$

These are numerically-valued G-power series of the function $y^*[f(x)]$. But the power series representation is unique according to Theorem 4.3.10, whence it follows that $y^*[P_k(\theta, x)] = y^*[P_k(x)]$ for every $x \in \mathfrak{D}$ and $y^* \in \mathfrak{Y}^*$, $k = 0, 1, 2, \cdots$. It follows that $P_k(x) \equiv P_k(\theta, x)$. But the series $\sum_0^\infty P_k(\theta, x)$ is strongly convergent in $\mathfrak{D}$, hence $\sum_0^\infty P_k(x)$ has the same property.

We come now to the question of the structure of the finitely open set of convergence of a G-power series. Theorem 4.4.2 has a bearing on this problem. We have further:

THEOREM 4.4.9. *The largest finitely open set in which a G-power series converges is a c-star about θ.*

PROOF. We assume the existence of a finitely open set of convergence. It is clear that the union of all such sets is a finitely open set of convergence and that it is the largest such set. It remains to prove that it is a c-star about θ. Denoting this largest set by $\mathfrak{G}_0[P_k]$, we assume that $x_0 \in \mathfrak{G}_0[P_k]$, $x_0 \neq \theta$. In the comments to Definition 4.3.2 it was observed that to every point of a finitely

open set there is a finitely open c-star about that point which is also in the set. Let $\mathfrak{C}^*(x_0) = x_0 + H$ be the c-star in $\mathfrak{G}_0[P_k]$ about x_0. By Theorem 4.4.2, $\mathfrak{G}_0[P_k]$ then contains the finitely open c-star $\mathfrak{C}^*(\theta) = \zeta x_0 + H$, $|\zeta| \leqq 1$. In particular, we see that $x_0 \in \mathfrak{G}_0[P_k]$ implies $\zeta x_0 \in \mathfrak{G}_0[P_k]$, for $|\zeta| \leqq 1$, that is, $\mathfrak{G}_0[P_k]$ is a c-star.

Definition 4.4.2. *A subset $\mathfrak{D}$ of $\mathfrak{X}$ is called c-convex if, Δ being any bounded open set of complex numbers and Γ its boundary, the assumption that $x + \Gamma h$ is contained in $\mathfrak{D}$ implies that $x + \Delta h$ lies in $\mathfrak{D}$.*

Theorem 4.4.10. *$\mathfrak{G}_0[P_k]$ is c-convex.*

Proof. Suppose that Δ is a bounded open set of complex numbers and Γ is its boundary; let x_0 and h_0 be such that the set $\mathfrak{S} \subset \mathfrak{G}_0[P_k]$ where $\mathfrak{S} = x_0 + \zeta h$, $\zeta \in \Gamma$. Since $\mathfrak{S}$ lies in a two-dimensional linear subspace and is compact there, Theorem 4.3.8 applies. Consequently there exist a finitely open c-star $\mathfrak{C}$ about θ and an ϵ, $0 < \epsilon < 1$, such that (i) $\mathfrak{S} \subset (1 - \epsilon)\mathfrak{C} \subset \mathfrak{C} \subset \mathfrak{G}_0[P_k]$ and (ii) $\sum M_k$ converges where $M_k = \sup \| P_k(x) \|$ for $x \in (1 - \epsilon)\mathfrak{C}$. Consider now the set H of all elements $h \in \mathfrak{X}$ such that $x_0 + \zeta h_0 + h \in (1 - \epsilon)\mathfrak{C}$ for every $\zeta \in \Gamma$. It is claimed that H is finitely open. To see this, intersect H by a finite-dimensional linear subspace $\mathfrak{X}_n$; let $h \in H \cap \mathfrak{X}_n = H_n$ and let $(1 - \epsilon)\mathfrak{C} \cap \mathfrak{X}_n = \mathfrak{C}_n$. Then $x_0 + \zeta h_0 + h \in \mathfrak{C}_n$ for every $\zeta \in \Gamma$. Suppose we could find a sequence $\{h_\nu\} \in \mathfrak{X}_n$ and a sequence $\{\zeta_\nu\} \in \Gamma$ such that $h_\nu \to h$ and $x_0 + \zeta_\nu h_0 + h_\nu \in \mathfrak{X}_n - \mathfrak{C}_n$. Without loss of generality we may assume that $\zeta_\nu \to \zeta_0 \in \Gamma$. We have then $x_0 + \zeta_\nu h_0 + h_\nu \to x_0 + \zeta_0 h_0 + h$, an element of the relatively open set $\mathfrak{C}_n$. This involves a contradiction and we see that for all large values of ν we must have $x_0 + \zeta_\nu h_0 + h_\nu \in \mathfrak{C}_n$. It follows that H_n is relatively open and that H is finitely open.

We now consider $P_k(x_0 + \zeta h_0 + h)$ for $h \in H$. But when $\zeta \in \Gamma$ we have $x_0 + \zeta h_0 + h \in (1 - \epsilon)\mathfrak{C}$ so that $\| P_k(x_0 + \zeta h_0 + h) \| \leqq M_k$. By the principle of the maximum, this inequality holds also for $\zeta \in \Delta$. It follows that $\sum \| P_k(x) \|$ converges uniformly for $x \in x_0 + \Delta h_0 + H$ which is a finitely open set. Hence $x_0 + \Delta h_0 + H \in \mathfrak{G}_0[P_k]$. In particular, $x_0 + \zeta h_0 \in \mathfrak{G}_0[P_k]$ for $\zeta \in \Delta$ and the theorem is proved.

Let $\mathfrak{D}$ be a finitely open set in $\mathfrak{X}$. We can always find a finitely open c-convex c-star about θ containing $\mathfrak{D}$ and if $\mathfrak{D} \neq \mathfrak{X}$, there are infinitely many such stars which form a partially ordered system under the relation of inclusion. The intersection of any finite number of such stars is a star with the same property; for an infinite system the intersection is still a c-convex c-star but may possibly fail to be finitely open. Let $\mathfrak{T}$ denote the intersection and $\mathfrak{J}$ the "finite interior" of $\mathfrak{T}$, that is, the set of all points of $\mathfrak{T}$ belonging to finitely open subsets of $\mathfrak{T}$. It is clear that $\mathfrak{D} \subset \mathfrak{J}$ and $\mathfrak{J}$ is finitely open. It is not so obvious that $\mathfrak{J}$ is also a c-convex c-star. To see this, suppose that $x_0 \in \mathfrak{J} \subset \mathfrak{T}$ so that $\zeta x_0 \in \mathfrak{T}$ for $|\zeta| \leqq 1$ and there is a finitely open set H containing θ such that $x_0 + H \subset \mathfrak{J}$. It follows that $\zeta x_0 + \zeta H \subset \mathfrak{T}$ for $|\zeta| \leqq 1$. But the left side of this inclusion is a finitely open set containing ζx_0; thus $\zeta x_0 \in \mathfrak{J}$ and $\mathfrak{J}$ is a c-star. The c-convexity is proved by the same type of argument. The assumption that $x_0 + \Gamma h_0 \subset \mathfrak{J}$ implies the existence of

a finitely open set H such that $x_0 + \Gamma h_0 + H \subset \mathfrak{Z}$ or $(x_0 + H) + \Gamma h_0 \subset \mathfrak{Z} \subset \mathfrak{T}$. Since $\mathfrak{T}$ is c-convex, this implies that $(x_0 + H) + \Delta h_0 \subset \mathfrak{T}$. But $x_0 + \Delta h_0 + H$ is a finitely open set containing $x_0 + \Delta h_0$ and thus belonging to $\mathfrak{Z}$. This proves the c-convexity of $\mathfrak{Z}$.

Thus to every system of finitely open c-convex c-stars containing $\mathfrak{D}$ there is a unique smallest set with the same properties, contained in all of them, namely the finite interior of the intersection of the sets in the system. By Zorn's lemma there is a minimal finitely open c-convex c-star containing $\mathfrak{D}$ and contained in all such stars. We denote this minimal star by $\mathfrak{C}[\mathfrak{D}]$. The two preceding theorems then imply

THEOREM 4.4.11. *If $\mathfrak{D}$ is a given finitely open set in $\mathfrak{X}$ and if a G-power series $\sum P_k(x)$ converges in $\mathfrak{D}$ then $\mathfrak{C}[\mathfrak{D}] \subset \mathfrak{G}_0[P_k]$.*

From this theorem it follows that the assumption made in the proof of Theorem 4.4.8 that $\mathfrak{D}$ is a c-star about θ is no restriction. Another consequence is the following

THEOREM 4.4.12. *If $f(x)$ is G-differentiable in a finitely open c-star $\mathfrak{D}$ about θ, then it may be continued as a G-differentiable function into all of $\mathfrak{C}[\mathfrak{D}]$.*

2. Analytic Functions

4.5. F-differentiability. In the G-theory developed in the preceding paragraph, the topology of the argument space $\mathfrak{X}$ played a subordinate role and practically the whole discussion could be carried through for an arbitrary complex linear space $\mathfrak{X}$. The topology of the range space $\mathfrak{Y}$ was more important, but the theory could probably be extended, for instance from (B)- to (F)-spaces, without too much effort. In the present paragraph we shall be concerned with F-differentiable functions and here the assumption that both spaces are of type (B) will be utilized more fully although undoubtedly weaker assumptions could be made without serious detriment to the theory. In passing from the G- to the F-theory, finitely open sets are replaced by open sets which are required for F-differentiability.

Definition 4.3.4 shows that an F-differentiable function is continuous and G-differentiable. The converse is also true and, in fact, continuity may be replaced by local boundedness or even continuity in the sense of Baire. The sharpest of these results will be proved later, but for the time being local boundedness provides a convenient working condition.

DEFINITION 4.5.1. *A function $f(x)$ on $\mathfrak{X}$ to $\mathfrak{Y}$ is said to be locally bounded in the open set $\mathfrak{D}$ if for every point a in $\mathfrak{D}$ there is a sphere $\mathfrak{S}_a$, $\| x - a \| \leqq r_a$, and a finite $M(a)$ such that $\| f(x) \| \leqq M(a)$ when $x \in \mathfrak{S}_a$.*

DEFINITION 4.5.2. *A function $f(x)$ on $\mathfrak{X}$ to $\mathfrak{Y}$, defined in the domain $\mathfrak{D}$, is said to be analytic in $\mathfrak{D}$ if it is single-valued, locally bounded, and G-differentiable in $\mathfrak{D}$.*

THEOREM 4.5.1. *If $f(x)$ is analytic in the domain $\mathfrak{D}$, then it is continuous and F-differentiable in $\mathfrak{D}$. It has variations of all orders which are continuous, F-differentiable functions of x and of all the increments $h_1, \cdots, h_n$, that is, $\delta^n f(x; h_1, \cdots, h_n)$ is an analytic function of x in $\mathfrak{D}$ for fixed $h_1, \cdots, h_n$ and a continuous multilinear form in $h_1, \cdots, h_n$ for fixed x in $\mathfrak{D}$. The Taylor expansion*

$$f(x + h) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n f(x; h)$$

is an F-power series in h.

PROOF. Since $f(x)$ is G-differentiable, it has variations of all orders which are G-differentiable functions of x and symmetric multilinear forms in the increments h_i. Further $f(x)$ may be expanded in a Taylor's series about every point x in $\mathfrak{D}$ and the assumption that $f(x)$ is locally bounded enables us to apply Theorem 4.3.7 with $\mathfrak{C}^*(a)$ replaced by $\mathfrak{S}_a$. This gives

$$\| \delta^n f(x; h) \| \leqq n! M(a) \quad \text{if} \quad \| x - a \| \leqq \tfrac{1}{2} r_a, \quad \| h \| \leqq \tfrac{1}{2} r_a,$$

so that $x + h \in \mathfrak{S}_a$. Since $\delta^n f(x; h)$ is homogeneous of degree n in h, we have consequently for all h

$$\| \delta^n f(x; h) \| \leqq n! M(a) (2/r_a)^n \| h \|^n \quad \text{if} \quad \| x - a \| \leqq \tfrac{1}{2} r_a.$$

This proves that $\delta^n f(x; h)$ is a continuous function of h, locally uniformly in x, and locally bounded with respect to x in $\mathfrak{D}$, uniformly in h in any finite domain. Combining these estimates with the Taylor expansion, we obtain

$$\| f(x + h) - f(x) \| \leqq \frac{2M(a) \| h \|}{r_a - 2 \| h \|},$$

$$\| f(x + h) - f(x) - \delta f(x; h) \| \leqq \frac{4M(a) \| h \|^2}{r_a(r_a - 2 \| h \|)}$$

for $\| x - a \| \leqq \frac{1}{2} r_a$, $\| h \| < \frac{1}{2} r_a$, so that $f(x)$ is continuous and F-differentiable in $\mathfrak{D}$. Since $\delta^n f(x; h)$, considered as a function of x for fixed h, is locally bounded and G-differentiable in $\mathfrak{D}$, the same argument shows that $\delta^n f(x; h)$ is continuous and F-differentiable with respect to x in $\mathfrak{D}$. This conclusion extends also to $\delta^n f(x; h_1, \cdots, h_n)$ which is the polar form of $\delta^n f(x; h)$ and hence linearly expressible in terms of the functions $\delta^n f(x; \epsilon_1 h_1 + \cdots + \epsilon_n h_n)$, $\epsilon_n = 0, 1$, all of which are continuous F-differentiable functions of x. The argument shows that $\delta^n f(x; h)$ is an analytic function of x in $\mathfrak{D}$; being a continuous function of h for fixed x, it is an F-power of degree n in h, and the Taylor series is an F-power series in h.

The G-theory asserts that the largest finitely open region of convergence of the Taylor series is a c-convex c-star about x. That the largest open region of convergence is also a c-convex c-star will follow from the discussion in section 4.7.

4.6. Properties of analytic functions. We shall prove some theorems about vector-valued analytic functions which are obvious analogues of classical theorems in the numerical case.

THEOREM 4.6.1. *An analytic function which vanishes in a sphere vanishes identically in its domain of analyticity.*

PROOF. Let $\mathfrak{D}$ be the domain in which $f(x)$ is supposed to be defined as an analytic function and suppose that $f(x) = \theta$ in the sphere $\mathfrak{S}_a : \| x - a \| < r$. Then formula (4.3.1) shows that $\delta^n f(a; x - a) = \theta$ for $\| x - a \| < r$ and hence for all x. From Theorem 4.3.6 it follows that $f(x) = \theta$ in the largest sphere $\mathfrak{S}$ with center at $x = a$ and contained in $\mathfrak{D}$. Now if b is any point in $\mathfrak{D}$ not in $\mathfrak{S}$, we may join the points a and b by a finite chain of open spheres in $\mathfrak{D}$, $\mathfrak{S}_0 = \mathfrak{S}$, $\mathfrak{S}_1, \cdots, \mathfrak{S}_n$, such that $\mathfrak{S}_k$ contains the center of $\mathfrak{S}_{k+1}$. Since $f(x) = \theta$ in $\mathfrak{S}_0$, the preceding argument shows that $f(x) = \theta$ in $\mathfrak{S}_1$ and hence, by induction, $f(b) = \theta$ and $f(x) \equiv \theta$ in $\mathfrak{D}$.

It should be observed that the conclusion is not necessarily valid if it is known merely that $f(x)$ vanishes in a point set having a limit point in $\mathfrak{D}$. In particular, the theory of linear bounded functionals provides examples of analytic functions of a vector variable capable of vanishing on a linear subspace of $\mathfrak{X}$ without vanishing identically. However, the theorem as formulated makes it possible to apply the classical process of analytic continuation, with obvious modifications, to the present class of analytic functions.

The convergence theorems of classical function theory extend, *mutatis mutandis*, to vector-valued analytic functions. We restrict ourselves to the analogs of well-known theorems of Vitali and Osgood. Both the assumptions and the conclusions differ somewhat from the classical prototypes, however; in particular, no information is obtained relating to uniform convergence of the given sequence.

THEOREM 4.6.2. *Let $\{f_k(x)\}$ be a sequence of functions analytic and locally uniformly bounded in a fixed domain $\mathfrak{D}$. If $\lim_{k\to\infty} f_k(x)$ exists in a sphere $\mathfrak{S}$ in $\mathfrak{D}$, then the limit $f(x)$ exists everywhere in $\mathfrak{D}$ and is analytic there. Further*

$$\delta^n f(x; h) = \lim_{k\to\infty} \delta^n f_k(x; h)$$

for every n and all x in $\mathfrak{D}$.

PROOF. The assumption of local uniform boundedness is understood in the following sense: to every point $x_0 \in \mathfrak{D}$ there is a sphere $\mathfrak{S}(x_0)$ and a finite positive quantity $M(x_0)$ such that $\| f_k(x) \| \leqq M(x_0)$ for $x \in \mathfrak{S}(x_0)$ and $k = 1, 2, 3, \cdots$. For the proof it is enough to show that $\lim_{k\to\infty} f_k(x)$ exists everywhere in $\mathfrak{D}$ and that the limit $f(x)$ is also a G-differentiable function. Since $\| f(x) \| \leqq M(x_0)$ for $x \in \mathfrak{S}(x_0)$, $f(x)$ is locally bounded and hence analytic. Since $f_k(x)$ is analytic in $\mathfrak{D}$, it is G-differentiable there, that is, for fixed $x \in \mathfrak{D}$, $h \in \mathfrak{X}$ the function $f_k(x + \zeta h)$ is holomorphic for $| \zeta | < \rho(x, h)$. If $x \in \mathfrak{S}$, there exists a $\rho_0(x, h)$ such that $x + \zeta h \in \mathfrak{S} \cap \mathfrak{S}(x)$ when $| \zeta | < \rho_0(x, h)$. For such values

of ζ we have $\|f_k(x + \zeta h)\| \leqq M(x)$ and $\lim_{k\to\infty} f_k(x + \zeta h) = f(x + \zeta h)$ and, by the Vitali Theorem 3.13.2, $f(x + \zeta h)$ is holomorphic in ζ. This implies that $f(x)$ is G-differentiable and hence analytic in $\mathfrak{S}$.

In order to extend this result to all of $\mathfrak{D}$ we use the classical machinery used in analytic continuation. Let a be the center of $\mathfrak{S}$, b any point of $\mathfrak{D} - \mathfrak{S}$. We then join a with b by a finite chain of interlaced spheres $\mathfrak{S}_0, \mathfrak{S}_1, \cdots, \mathfrak{S}_N$, such that the center of $\mathfrak{S}_\nu$ is a_ν, $a_0 = a$, $a_N = b$, $a_{\nu+1} \in \mathfrak{S}_\nu$, $\mathfrak{S}_0 \subset \mathfrak{S}$ and $\mathfrak{S}_\nu \subset \mathfrak{D}$. We may assume that $\mathfrak{S}_\nu \subset \mathfrak{S}(a_\nu)$ without loss of generality, since otherwise we have merely to intercalate more and smaller spheres having their centers on the polygonal line with vertices at $a_0, a_1, \cdots, a_N$ which line is a compact set. Let ρ_ν be the radius of $\mathfrak{S}(a_\nu)$ and put $\mathfrak{E}_\nu = \mathfrak{S}_0 \cup \mathfrak{S}_1 \cup \cdots \cup \mathfrak{S}_\nu$, $0 \leqq \nu \leqq N$. If $M = \max M(a_\nu)$, we have $\|f_k(x)\| \leqq M$ for all x in $\mathfrak{E}_N$. Suppose that we have shown the existence and analyticity of $f(x)$ in $\mathfrak{E}_\nu$ and let us show it in $\mathfrak{E}_{\nu+1} = \mathfrak{E}_\nu \cup \mathfrak{S}_{\nu+1}$. For fixed h the sequence $\{f_k(a_{\nu+1} + \zeta h)\}$ is made up of functions which are holomorphic and in norm not exceeding M in the circle $|\zeta| < \rho_{\nu+1}/\|h\|$ and the sequence converges to $f(a_{\nu+1} + \zeta h)$ when $|\zeta| < [\rho_\nu - |a_{\nu+1} - a_\nu\|]/\|h\|$. By the Vitali Theorem 3.13.2 the sequence then converges also for $|\zeta| < \rho_{\nu+1}/\|h\|$ and the limit $f(a_{\nu+1} + \zeta h)$ is holomorphic also in the larger circle. It follows in particular that $\lim_{k\to\infty} f_k(x) = f(x)$ exists for x in $\mathfrak{S}_{\nu+1}$ and hence everywhere in $\mathfrak{E}_{\nu+1}$. It remains to prove that $f(x)$ is analytic in $\mathfrak{S}_{\nu+1}$. For this purpose it is enough to observe that for fixed $h \in \mathfrak{X}$ and $x_0 \in \mathfrak{S}_{\nu+1}$, the sequence $\{f_k(x_0 + \zeta h)\}$ is made up of functions holomorphic and uniformly bounded in some circle $|\zeta| < \rho_0(x_0, h)$ where it converges to $f(x_0 + \zeta h)$. The limit is then holomorphic in the same circle whence it follows that $f(x)$ is G-differentiable and hence analytic in $\mathfrak{S}_{\nu+1}$. The argument also shows that the process furnishes the analytic continuation of $f(x)$ from $\mathfrak{E}_\nu$ to $\mathfrak{E}_{\nu+1}$. Since $f(x)$ is known to be analytic in $\mathfrak{E}_0$, it follows that the induction argument applies so that $f(x)$ is analytic in $\mathfrak{E}_N$ and hence ultimately everywhere in $\mathfrak{D}$. The bounded convergence of $f_k(x_0 + \zeta h)$ to $f(x_0 + \zeta h)$ for $|\zeta| < \rho_0(x_0, h)$ also implies convergence of the derivatives, whence it follows that

$$\lim_{k\to\infty} \delta^n f_k(x; h) = \delta^n f(x; h)$$

for all n and all x in $\mathfrak{D}$. This completes the proof.

THEOREM 4.6.3. *If the functions $f_k(x)$ are analytic in a fixed domain $\mathfrak{D}$ and if $\lim_{k\to\infty} f_k(x) = f(x)$ exists for every x in $\mathfrak{D}$, then every open subset of $\mathfrak{D}$ contains a domain in which $f(x)$ is analytic.*

REMARK. In other words, the domains of analyticity of $f(x)$ are everywhere dense in $\mathfrak{D}$. There exists at least one such domain; there may be infinitely many, the number being limited only by the cardinal number of non-overlapping domains in the space $\mathfrak{X}$. The analytic functions defined in the different domains are not necessarily related by analytic continuation.

PROOF. Let $\mathfrak{G}$ be an open subset of $\mathfrak{D}$ and let $\mathfrak{F}_n$ be the closed subset of $\mathfrak{G}$ in which $\|f_k(x)\| \leqq n$ for all k simultaneously. Since $\lim_{k\to\infty} f_k(x) = f(x)$ exists

everywhere in $\mathfrak{D}$, each point $x \in \mathfrak{G}$ belongs to $\mathfrak{F}_n$ for $n > n_x$. We have $\mathfrak{F}_n \subset \mathfrak{F}_{n+1}$ and $\lim \mathfrak{F}_n = \mathfrak{G}$. Consequently there is an integer N such that $\mathfrak{F}_N$ contains a sphere $\mathfrak{S}$ so that $\| f_k(x) \| \leqq N$ for $x \in \mathfrak{S}$ and all k. The functions $f_k(x)$ being analytic and uniformly bounded in $\mathfrak{S}$, the existence of $\lim_{k\to\infty} f_k(x) = f(x)$ in $\mathfrak{S}$ implies that $f(x)$ is analytic in $\mathfrak{S}$ by virtue of the preceding theorem. This completes the proof.

We conclude the discussion by extending the principle of the maximum.

THEOREM 4.6.4. *Let $f(x)$ be analytic in the domain $\mathfrak{D}$ and continuous in its closure $\bar{\mathfrak{D}}$. If* $\sup \| (f(x) \| = M$ *for* $x \in \bar{\mathfrak{D}} - \mathfrak{D}$, *then either* $\| f(x) \| < M$ *in* $\mathfrak{D}$ *or else* $\| f(x) \| \equiv M$ *in* $\mathfrak{D}$.

PROOF. Suppose $a \in \mathfrak{D}$ and $\| f(a) \| = \alpha \geqq M$. Let $h \in \mathfrak{X}$ be fixed but arbitrary and consider the linear manifold $x = a + \zeta h$. It intersects $\mathfrak{D}$ in a relatively open set corresponding to an open set Δ in the complex plane. Let Δ_0 be the component of Δ which contains $\zeta = 0$. Then $f(a + \zeta h)$ is a holomorphic function of ζ in Δ_0; it is continuous in $\bar{\Delta}_0$ and $\sup \| f(a + \zeta h) \| \leqq M$ for $\zeta \in \bar{\Delta}_0 - \Delta_0$. By the maximum principle of section 3.12, this implies $\| f(a + \zeta h) \| \leqq M$ for $\zeta \in \Delta_0$. It follows that $\| f(a) \| = \alpha = M$. But then the principle asserts that $\| f(a + \zeta h) \| = M$ everywhere in Δ_0. In this argument h was arbitrary. Consequently the asumption that $\| f(a) \| > M$ is self-contradictory and $\| f(a) \| = M$ requires that $\| f(x) \| = M$ for every x in $\mathfrak{D}$ which may be joined with a by a straight line segment in $\mathfrak{D}$. This in turn implies that $\| f(x) \| = M$ for every x in $\mathfrak{D}$ which may be joined with a by a polygonal line in $\mathfrak{D}$. Since all points of $\mathfrak{D}$ have this property, we conclude that $\| f(x) \| \equiv M$ in $\mathfrak{D}$ if equality holds at a single point of $\mathfrak{D}$.

It should be observed that it is not permitted to conclude from $\| f(a) \| = M$ that $f(x)$ is a constant of norm M. As a counter example we may take $\mathfrak{X} = \mathfrak{Y} = Z_2$ with $x = \zeta = (\zeta_1, \zeta_2)$, $\| \zeta \| = \max (| \zeta_1 |, | \zeta_2 |)$, and $f(\zeta) \doteq (1, \zeta_2)$. This is a linear continuous and hence also analytic function of ζ and $\| f(\zeta) \| = \max (1, | \zeta_2 |)$. Here $\| f(\zeta) \| = 1$ when $\| \zeta \| \leqq 1$, but $f(\zeta)$ is not a constant.

4.7. F-power series. Let $\{P_k(x)\}$ be a given sequence of F-powers, the degree of $P_k(x)$ being k, and consider the F-power series $\sum_0^\infty P_k(x)$. This is of course also a G-power series so the results of section 4.4 apply. Owing to the continuity of the terms, sharper results may be proved, however.

THEOREM 4.7.1. *If $\mathfrak{C}[P_k]$ is the region of convergence of an F-power series and if* $\mathfrak{G}[P_k] \equiv \operatorname{Int} \mathfrak{C}[P_k]$ *is non-void, then $\mathfrak{G}[P_k]$ is a c-convex c-star about θ.*

PROOF. Suppose that $\mathfrak{G}[P_k]$ is non-vacuous and contains the point x_0. There is then also an open c-star $\{x_0 + H\}$ about x_0 contained in $\mathfrak{G}[P_k]$ and by Theorem 4.4.2 the series converges in the set $\{\zeta x_0 + H\}$, $| \zeta | \leqq 1$, which is an open c-star about θ. It follows that $\mathfrak{G}[P_k]$ is also an open c-star about θ. The c-convexity requires a more elaborate argument. Suppose that there is an open set Δ in the complex plane with boundary Γ such that for some x_0 and h_0

the point set $\{x_0 + \zeta h_0\}$, $\zeta \in \Gamma$, belongs to $\mathfrak{G}[P_k]$. This implies that for every $\zeta_0 \in \Gamma$ there is a positive $\rho(\zeta_0)$ such that $x_0 + \zeta_0 h_0 + h \in \mathfrak{G}[P_k]$ if $\| h \| < \rho(\zeta_0)$. The set Γ being compact, $\inf \rho(\zeta_0) \equiv \rho > 0$. Thus $x_0 + h + \zeta h_0 \in \mathfrak{G}[P_k]$ when $\zeta \in \Gamma$, $\| h \| < \rho$. Now consider the series $\sum_0^\infty P_k[\zeta_1(x_0 + h) + \zeta_2 h_0]$, h fixed, $\| h \| < \rho$. This is a series of homogeneous polynomials in (ζ_1, ζ_2) converging in some open set Δ_0 of Z_2 containing the set $(1, \Gamma)$. By Theorem 3.14.2 we can find a neighborhood Δ_1 of $(1, \Gamma)$ in Δ_0 such that $\sum_k M_k$ converges where M_k is the supremum of $\| P_k[\zeta_1(x_0 + h) + \zeta_2 h_0] \|$ for (ζ_1, ζ_2) in Δ_1. In particular, $\| P_k(x_0 + h + \zeta h_0) \| \leqq M_k$, $\zeta \in \Gamma$, and by the principle of the maximum the same inequality is true for $\zeta \in \Delta$. Thus the series $\sum \| P_k(x) \|$ converges for every x of the form $x_0 + h + \zeta h_0$, $\zeta \in \Delta$, $\| h \| < \rho$, and these points form an open set containing the set $\{x_0 + \zeta h_0\}$, $\zeta \in \Delta$. Hence the latter set belongs to $\mathfrak{G}[P_k]$ and the c-convexity is proved.

It is natural to expect that an F-power series should converge to an analytic function in $\mathfrak{G}[P_k]$, but this simple proposition is by no means easy to prove. The first stage of the proof is given in

THEOREM 4.7.2. *If $\mathfrak{G}[P_k]$ is non-vacuous, it contains a non-vacuous open c-star about θ in which the series converges uniformly to an analytic function.*

PROOF. If $\mathfrak{G}[P_k]$ is non-void, the sum of the series, $f(x)$ say, is a G-differentiable function of x in $\mathfrak{G}[P_k]$ by Theorem 4.4.6 and $f(x)$ may be expanded in a convergent Taylor series about each point of $\mathfrak{G}[P_k]$. In particular, the Maclaurin series is seen to be identical to the given F-power series, that is, $\delta^k f(\theta; x) = k!\, P_k(x)$ for all k and all x. We can then appeal to Theorem 4.3.8 which asserts the existence of a finitely open c-star $\mathfrak{C}$ about θ such that $\sum_k M_k$ converges where M_k is the supremum of $\| P_k(x) \|$ for x in $\mathfrak{C}$. $P_k(x)$ being continuous, we may replace $\mathfrak{C}$ by its closure $\bar{\mathfrak{C}}$ without changing the value of the supremum. It is desired to show that $\bar{\mathfrak{C}}$ contains an open sphere. Let $\mathfrak{C}_n$ denote the set of all points x such that $(1/n)x \in \mathfrak{C}$, $n = 2, 3, \cdots$. Every point $x \in \mathfrak{X}$ belongs to some $\bar{\mathfrak{C}}_n$ since $\mathfrak{C}$ is finitely open and contains θ. It follows that $\bar{\mathfrak{C}}_n \subset \bar{\mathfrak{C}}_{n+1}$ and $\lim \bar{\mathfrak{C}}_n = \mathfrak{X}$. This shows that some set $\bar{\mathfrak{C}}_n$ contains an open sphere since $\mathfrak{X}$ cannot be of the first category in itself. It follows that $\bar{\mathfrak{C}}$ contains an open sphere, $\mathfrak{S}$: $\| x - a \| < \rho$ say. We have then $\| P_k(x) \| \leqq M_k$ for $x \in \mathfrak{S}$. By Lemma 4.4.1 the same inequality is valid in the open c-star $\mathfrak{C}_0 : \{\zeta a + h\}$, $| \zeta | \leqq 1$, $\| h \| < \rho$. In $\mathfrak{C}_0$ the F-power series converges uniformly to $f(x)$; the terms being continuous, $f(x)$ is also continuous in $\mathfrak{C}_0$ and consequently analytic there.

We note in passing the following theorem, the proof of which is left to the reader:

THEOREM 4.7.3. *If the terms of an F-power series are locally uniformly bounded in an open set $\mathfrak{D}$, then the series converges locally uniformly in $\mathfrak{D}$ and vice versa. The sum of the series is analytic in $\mathfrak{D}$.*

We come now to the main theorem:

THEOREM 4.7.4. *The sum of an F-power series is analytic in* $\mathfrak{G}[P_k]$

PROOF. We already know that the sum $f(x)$ is analytic in some neighborhood of $x = \theta$. We now take $a \in \mathfrak{G}[P_k]$, $a \neq \theta$, and consider the Taylor series

$$f(a + h) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n f(a; h)$$

which is a G-power series in h by Theorem 4.3.10. As such it converges in the largest finitely open c-convex c-star about a contained in the finite interior of $\mathfrak{G}[P_k]$. In particular, it will converge in a non-void open c-star about a, that is, it converges for h in a non-void open c-star about θ. We shall prove that this series is actually an F-power series in h. To this end we note that

$$\delta^n f(a; h) = \sum_{k=n}^{\infty} \delta^n P_k(a; h),$$

the convergence of the series being ensured by Theorem 4.4.5 for all values of n. Here $\delta^n P_k(a; h)$ is a homogeneous polynomial in h of degree n. It was shown in the proof of Theorem 4.3.9 that $\| \delta^n P_k(a; h) \| \leqq M_k n! \| a \|^{k-n} \| h \|^n$ so that the terms are F-powers in h of fixed degree n. By Theorem 4.2.9 the sum of the series $\delta^n f(a; h)$ is then either identically θ or else an F-power in h of degree n. Thus the Taylor series for $f(a + h)$ is actually an F-power series in h. The series being convergent in an open set containing $h = \theta$, Theorem 4.7.2 asserts that it converges uniformly in some neighborhood of $h = \theta$. Hence $f(a + h)$ is an analytic function of h for small values of $\| h \|$, that is, $f(x)$ is analytic everywhere in $\mathfrak{G}[P_k]$.

THEOREM 4.7.5. *If* $a \in \mathfrak{G}[P_k]$, *then there exists an open c-star* $\mathfrak{C}$ *about* θ *containing* a, *such that* $\sum_k M_k$ *converges where* M_k *is the supremum of* $\| P_k(x) \|$ *in* $\mathfrak{C}$.

PROOF. This is the analogue for F-differentiable functions of Theorem 4.3.8 and the same type of proof applies. Since $f(x)$ is now continuous in $\mathfrak{G}[P_k]$, the function $\varphi(x) = \max \| f(\zeta x) \|$, $| \zeta | = 1$, is also a continuous function in $\mathfrak{G}[P_k]$. The set $\mathfrak{C}_0$ of points x such that $\varphi(x) < \varphi(a) + 1$ is an open c-star about θ containing $x = a$. If ϵ is a small positive number, $\mathfrak{C} = (1 - \epsilon)\,\mathfrak{C}_0$ still contains $x = a$ and is an open c-star about θ. From Theorem 4.3.7 we get $\| \delta^k f(\theta; x) \| \leqq k! \, [1 + \varphi(a)](1 - \epsilon)^k$ for x in $\mathfrak{C}$, so that $M_k \leqq [1 + \varphi(a)](1 - \epsilon)^k$ and the convergence of the series $\sum M_k$ is obvious.

COROLLARY. *The F-power series* $\sum_k P_k(x)$ *converges locally uniformly in* $\mathfrak{G}[P_k]$.

The preceding theorems give a reasonably good qualitative description of $\mathfrak{G}[P_k]$. We can obtain some quantitative results related to the function $\rho(u)$ of Definition 4.4.1.

Definition 4.7.1. *Let* $\rho_a = \inf \rho(u)$, $\| u \| = 1$, *and put*

$$\rho_u = 1/\mu_u, \quad \mu_u = \limsup_{n\to\infty} \| P_n \|^{1/n}.$$

We call ρ_a *the radius of absolute and* ρ_u *the radius of uniform convergence of* $\sum_k P_k(x)$.

We recall that $\| P_n \|$ is the norm of $P_n(x)$ in the sense of Definition 4.2.4 and is finite if and only if $P_n(x)$ is an F-power. For a true G-power series we have always $\rho_u = 0$. The justification of the terminology is given by:

Theorem 4.7.6. *For the F-power series* $\sum_0^\infty P_n(x)$ *we have* $0 < \rho_u \leqq \rho_a$ *if and only if* $\mathfrak{S}[P_k]$ *is non-void. The series is absolutely convergent for* $\| x \| < \rho_a$; *this is the largest open sphere with center at* θ *contained in* $\mathfrak{S}[P_k]$. *On every spherical surface* $\| x \| = \rho$ *with* $\rho_a < \rho$ *there are points where the series diverges. It is uniformly convergent for* $\| x \| < (1 - \epsilon)\rho_u$, $\epsilon > 0$, *and fails to converge uniformly on any spherical surface* $\| x \| = \rho$ *with* $\rho_u < \rho$.

Proof. Suppose that $\rho_a > 0$. Then

$$\limsup_{n\to\infty} \| P_n(x) \|^{1/n} = \| x \|/\rho(u) \leqq \| x \|/\rho_a,$$

so that the series converges in the sphere $\| x \| < \rho_a$; $\mathfrak{S}[P_k]$ is non-void and contains the sphere. On the other hand, from the definition of the infimum it follows that for every $\epsilon > 0$ there is a point u on the unit sphere such that $\rho(u) < \rho_a + \epsilon$. Hence if $| \alpha | > \rho_a + \epsilon$, $\limsup_{n\to\infty} \| P_n(\alpha u) \|^{1/n} > 1$ and the series diverges for $x = \alpha u$. Thus every spherical surface $\| x \| = \rho$ with $\rho > \rho_a$ contains infinitely many points where the series diverges because the terms do not tend to zero. It follows that $\| x \| < \rho_a$ is the largest open sphere with center at θ contained in $\mathfrak{S}[P_k]$. The converse follows from Theorem 4.4.1.

If $\rho_u > 0$ and $\| x \| < (1 - \epsilon)\rho_u$, then

$$\| P_n(x) \| \leqq \| P_n \| \, \| x \|^n \leqq [\rho_u(1 - \epsilon/2)]^{-n}[\rho_u(1 - \epsilon)]^n < (1 - \epsilon/2)^n$$

for $n \geqq n(\epsilon)$ and the uniform convergence of the series for $\| x \| < (1 - \epsilon)\rho_u$ is evident. On the other hand, the definition of the supremum implies that for every $\epsilon > 0$ the inequality $\| P_n \| \geqq [\rho_u(1 - \epsilon)]^{-n}$ holds for infinitely many values of n. For each such n we can find a u_n on the unit sphere such that $\| P_n(u_n) \| \geqq [\rho_u(1 - 2\epsilon)]^{-n}$ and if ρ is given, $\rho_u < \rho$, ϵ may be chosen so small that

$$\| P_n(\rho u_n) \| = \rho^n \| P_n(u_n) \| > \rho^n[\rho_u(1 - 2\epsilon)]^{-n} = A^n$$

where $A > 1$. This implies that the series cannot converge uniformly on the spherical surface $\| x \| = \rho$ when $\rho > \rho_u$. Theorem 4.7.2 shows that $\rho_u > 0$ whenever $\rho_a > 0$. This completes the proof.

We may very well have $\rho_u < \rho_a$. This is shown by the following modification of the example used in section 4.4. We take $\mathfrak{X} = (l_2)$, $\mathfrak{Y} = Z_1$ and

$$f(x) = \sum_1^{\infty} (\alpha_n)^n, \qquad x = \{\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots\}.$$

This is an F-power series and a simple computation shows that $\rho_u = 1$ and $\rho_a = \infty$.

4.8. Further applications of Baire continuity.

We now return to the problem of deciding when a G-differentiable function is actually F-differentiable. The earliest solution of this problem appears to be due to A. E. Taylor (1937) who showed that continuity is sufficient. We replaced this assumption by local boundedness in section 4.5; the much weaker condition of continuity in the sense of Baire, found by Max Zorn, will now be considered. For the proof we shall need some simple lemmas which we state without proof.

LEMMA 4.8.1. *If $f(x)$ and $g(x)$ are Baire continuous, so are $\alpha f(x) + \beta g(x)$ and $f(\alpha x + a)$, where α, β, and a are fixed, $\alpha, \beta \in Z_1$, $a \in \mathfrak{X}$.*

LEMMA 4.8.2. *If $f(x)$ is Baire continuous for $\| x - a \| < \rho$ and $F(x) = f(x)$ or θ according as $\| x - a \| < \rho$ or $\geqq \rho$, then $F(x)$ is continuous in the sense of Baire for all x.*

THEOREM 4.8.1. *If $f(x)$ is defined as a G-differentiable function, continuous in the sense of Baire, in an open set $\mathfrak{D}$, then $f(x)$ is F-differentiable and hence analytic in $\mathfrak{D}$.*

PROOF. The basic idea of the proof is to show that the Baire continuity of $f(x)$ with respect to x extends to the variations $\delta^n f(x; h_1, \cdots, h_n)$ with respect to all $(n + 1)$ variables. Since these variations are symmetric n-linear forms of $h_1, \cdots, h_n$, Baire continuity implies ordinary continuity with respect to the increments by virtue of Theorem 4.2.8; this in turn implies that $\delta^n f(x; h)$ is an F-power in h and the Taylor series is an F-power series with a positive radius of absolute convergence so that $f(x + h)$ is analytic in h for small $\| h \|$ and $x \in \mathfrak{D}$. This makes $f(x)$ analytic in $\mathfrak{D}$.

We start with the first variation. Let $a \in \mathfrak{D}$ be fixed and let $\mathfrak{S}$: $\| x - a \| < \rho$ be a sphere in $\mathfrak{D}$. We define

$$F(x) = \begin{cases} f(x), & x \in \mathfrak{S}, \\ \theta, & x \in \mathfrak{X} - \mathfrak{S}; \end{cases}$$

$$F_n(x, h) = n\left[F\left(x + \frac{1}{n}h\right) - F(x)\right], \qquad n = 1, 2, 3, \cdots.$$

The functions are continuous in the sense of Baire with respect to the arguments involved by Lemmas 4.8.1 and 4.8.2. Further

$$\lim_{n\to\infty} F_n(x, h) = \delta f(x; h), \qquad x \in \mathfrak{S},\ h \in \mathfrak{X}.$$

Theorem 4.2.7 then implies that $\delta f(x; h)$ is Baire continuous with respect to x (in $\mathfrak{S}$ to start with but the extension to all of $\mathfrak{D}$ is immediate) and $h \in \mathfrak{X}$. As already observed, this makes $\delta f(x; h)$ continuous in h for x fixed in $\mathfrak{D}$. We can now apply the same type of argument to $\delta f(x; h_1)$. It is a G-differentiable, Baire continuous function of x in $\mathfrak{D}$ and a continuous function of h_1. It follows that $\delta^2 f(x; h_1, h_2)$ is continuous in the sense of Baire with respect to all three variables and hence continuous in the ordinary sense with respect to h_1 and h_2. In the same manner we prove that all variations $\delta^n f(x; h_1, \cdots, h_n)$ are continuous in the sense of Baire with respect to all variables and hence continuous with respect to $h_1, \cdots, h_n$ so that $\delta^n f(x; h)$ is also a continuous function of h for $x \in \mathfrak{D}$. The Taylor series

$$f(x + h) = \sum_0^\infty \frac{1}{n!} \delta^n f(x; h), \qquad x \in \mathfrak{D},$$

converges in a neighborhood of $h = \theta$ and is an F-power series in h. By Theorem 4.7.3 it is then an analytic function of h in this neighborhood of $h = \theta$. Hence $f(x)$ is analytic in $\mathfrak{D}$ and the theorem is proved.

This method appears to be quite powerful. With its aid Max Zorn has proved the following theorem which solves a problem which has been outstanding for some time.

THEOREM 4.8.2. *If $f(x_1, x_2)$ is defined on $\mathfrak{X} \times \mathfrak{X}$ to $\mathfrak{Y}$ and is F-differentiable with respect to each variable separately, then it is F-differentiable with respect to he pair (x_1, x_2). The norm in $\mathfrak{X} \times \mathfrak{X}$ is defined to be, for instance, $\| (x_1, x_2) \| = \| x_1 \| + \| x_2 \|$.*

CHAPTER V

ANALYSIS IN BANACH ALGEBRAS

5.1. Orientation. In the case of a Banach algebra we may consider functions (i) on scalars to the algebra, (ii) on the algebra to itself (or to another algebra), and (iii) on the algebra to scalars. The theory of functions of type (i) differs from the theory developed in Chapter III only by the added freedom resulting from the presence of products. For this reason we shall not devote much time to general questions, but concentrate the attention on a study of the *resolvent*, that is, the inverse of $\lambda e - x$, considered first as a function of λ and later also of x. This is a central problem in the theory of linear transformations in which case $x = T$ is an element of a space of endomorphisms of a given (B)-space. This problem is also basic in our approach to functions of type (ii) where we are primarily interested in *the problem of extending analytic scalar functions from the scalar field to the algebra.* Corresponding problems for the case in which the (B)-algebra does not have a unit element are solved in Chapter XXII. Among functions of type (iii) we consider only *linear bounded multiplicative functionals*, assuming multiplication to be commutative. These functionals are intimately connected with the spectral theory; their relations to functions on maximal ideals will be brought out in Chapter XXII. The latter chapter contains numerous complements of the analytical theory of (B)-algebras as well as the algebraic theory and should be consulted in connection with the reading of the present chapter.

There are five paragraphs: *Regular and Singular Elements*, *Functions on Scalars to the Algebra*, *Spectral Theory*, *Functions on the Algebra to Itself*, and *Functions on the Algebra to Scalars.* References are to be found at the end of each paragraph.

1. Regular and Singular Elements

5.2. Regular elements. Throughout this chapter the symbol $\mathfrak{B}$ denotes a complex Banach algebra having a unit element e. Multiplication is noncommutative unless the contrary is explicitly stated. All statements regarding limits and convergence are to be understood in terms of the normed metric of the space $\mathfrak{B}$.

There is no mention of inverses in our postulates, but it is a priori evident that some elements have inverses. The distinction between elements which have inverses and those which do not is fundamental in this theory.

Definition 5.2.1. *An element x is said to be regular if there is an element x^{-1}, called the inverse of x, such that $xx^{-1} = x^{-1}x = e$. A non-regular element is called singular. An element y (z) is said to be a right (left) inverse of x if $xy = e$ ($zx = e$).*

An element may have any number of either right inverses or left inverses. However, if $xy = zx = e$, then $zxy = ze$ and $y = z$. *Thus in an associative algebra the existence of a right inverse implies that the element has either no left inverse or the right inverse is also a left inverse and the element is regular.* In non-associative systems this is not necessarily true. If $\mathfrak{B}$ has no divisors of zero (see section 5.4), right and left inverses are also unique when they exist.

The distribution of regular elements is an important question to which the two following theorems give an answer.

Theorem 5.2.1. *Every element x in the open sphere $\| x - e \| < 1$ is regular and for such an x*

$$x^{-1} = e + \sum_{n=1}^{\infty} (e - x)^n. \tag{5.2.1}$$

Proof. The series is absolutely convergent so it defines an element of $\mathfrak{B}$ and multiplication by $x = e - (e - x)$ on right or left gives e.

Theorem 5.2.2. *The regular elements form an open set in $\mathfrak{B}$.*

Proof. We denote the set of regular elements by $\mathfrak{G}$ and shall show that if $x_0 \in \mathfrak{G}$, then the elements in the open sphere $\| x - x_0 \| < \| x_0^{-1} \|^{-1}$ also belong to $\mathfrak{G}$. Formally

$$\begin{aligned} x^{-1} &= [x_0 - (x_0 - x)]^{-1} = x_0^{-1}[e - (x_0 - x)x_0^{-1}]^{-1} \\ &= x_0^{-1} + x_0^{-1} \sum_{n=1}^{\infty} [(x_0 - x)x_0^{-1}]^n. \end{aligned} \tag{5.2.2}$$

This series, however, is absolutely convergent when $\| x - x_0 \| < \| x_0^{-1} \|^{-1}$ and multiplication by $x = x_0 - (x_0 - x)$ on right or left gives e. Thus the sum of the formal series is actually x^{-1} so the assertion is proved. Moreover, the series shows that

$$\| x^{-1} - x_0^{-1} \| \leqq \| x_0^{-1} \|^2 \| x - x_0 \| [1 - \| x - x_0 \| \| x_0^{-1} \|]^{-1}.$$

Hence:

Theorem 5.2.3. *The inverse x^{-1} is a continuous function of x in $\mathfrak{G}$.*

It should be noted that the property of being regular or being singular is not invariant under extensions and contractions of the algebra. Extensions preserve regularity but contractions may not; for singular elements the situation is reversed. We shall see later (section 5.13) that some elements may remain singular under all possible extensions of the algebra.

5.3. The maximal group. We return to the set $\mathfrak{G}$ of regular elements in $\mathfrak{B}$. It is clear that if $x, y \in \mathfrak{G}$, so does xy, and $(xy)^{-1} = y^{-1}x^{-1}$. Further, if $x \in \mathfrak{G}$, so does x^{-1}. Since $e \in \mathfrak{G}$ and the associative law holds in $\mathfrak{B}$, we see that $\mathfrak{G}$ is a group. We call $\mathfrak{G}$ the *maximal group* of $\mathfrak{B}$ since every other group in $\mathfrak{B}$ having e as its unit element must be a subgroup of $\mathfrak{G}$.

Being an open set, $\mathfrak{G}$ is the union of disjoint maximal open connected sets, $\mathfrak{G} = \bigcup_\alpha \mathfrak{G}_\alpha$, the *components* of $\mathfrak{G}$. *The component $\mathfrak{G}_1$ containing e is called the principal component or the kernel of* $\mathfrak{G}$. It will be shown in the Appendix [Theorem 22.4.2] that *in the commutative case $\mathfrak{G}$ has either a single component or else infinitely many.* In the latter case the number of components need not be countably infinite.

THEOREM 5.3.1. *The inverse of x is an analytic function of x in the sense of Definition* 4.5.2 *in each of the components of* $\mathfrak{G}$.

This is an immediate consequence of formula (5.2.2) and Definition 4.5.2.

THEOREM 5.3.2. *$\mathfrak{G}_1$ is an invariant subgroup of $\mathfrak{G}$ as well as the maximal connected subgroup of* $\mathfrak{G}$.

PROOF. Let a be a fixed but arbitrary element of $\mathfrak{G}$ and consider the mappings $y = ax$, $z = xa$. Both are homeomorphisms of $\mathfrak{B}$ onto itself and, in particular, $\mathfrak{G}$ is mapped onto itself. Since open sets go into open sets, each component of $\mathfrak{G}$ is mapped onto a component of $\mathfrak{G}$ in a one-to-one manner. If $a \in \mathfrak{G}_1$, then $a\mathfrak{G}_1 = \mathfrak{G}_1 a = \mathfrak{G}_1$, since both transformations map e on a. It follows that $ab \in \mathfrak{G}_1$ if a and $b \in \mathfrak{G}_1$. The inverse transformations also take components into components. In particular, if $a \in \mathfrak{G}_1$, then $a^{-1}\mathfrak{G}_1 = \mathfrak{G}_1 a^{-1} = \mathfrak{G}_1$, since both transformations take a into e. Thus, if $a \in \mathfrak{G}_1$, so does a^{-1}. Hence $\mathfrak{G}_1$ is a subgroup of $\mathfrak{G}$.

Finally, if a is any element of $\mathfrak{G}$, the mapping $u = a^{-1}xa$ takes components onto components. In particular, it leaves $\mathfrak{G}_1$ invariant since e is invariant. Hence $\mathfrak{G}_1$ is an invariant or a normal subgroup of $\mathfrak{G}$. It is also a maximal connected subgroup of $\mathfrak{G}$ since any larger subgroup must be disconnected. This completes the proof, but the result may be further strengthened as follows:

THEOREM 5.3.3. *$\mathfrak{G}_1$ is the only open connected subgroup of* $\mathfrak{G}$.

PROOF. Suppose that $\mathfrak{H}$ is an open connected subgroup of $\mathfrak{G}$. It is consequently a subgroup of $\mathfrak{G}_1$. If a is any element of $\mathfrak{G}_1$, then $a\mathfrak{H}$ is a left coset of $\mathfrak{H}$ with respect to $\mathfrak{G}_1$. As the homeomorphic image of an open set, $a\mathfrak{H}$ is also open. It follows that the union of any system of left (right) cosets of $\mathfrak{H}$ is open. Since the complement of $\mathfrak{H}$ with respect to $\mathfrak{G}_1$ is the union of left cosets of $\mathfrak{H}$, it will be open. Consequently, $\mathfrak{H}$ is both open and closed in $\mathfrak{G}_1$, whence it follows that $\mathfrak{H} = \mathfrak{G}_1$.

We observe that the components of $\mathfrak{G}$ are the elements of the *quotient group (factor group)* $\mathfrak{G}/\mathfrak{G}_1$.

For further properties of groups in $\mathfrak{B}$ see Chapter XXII.

5.4. Singular elements. The set $\mathfrak{F}$ of singular elements in $\mathfrak{B}$ is *closed* and contains at least the zero element. If $\mathfrak{B}$ is a field, θ is the only singular element. We note further that $\mathfrak{F}$ is *connected* because, if x is singular so is αx, and x is connected with the zero element by elements in $\mathfrak{F}$.

We note at this juncture some special types of elements which may occur in a (B)-algebra, all of which are singular and remain singular under arbitrary extensions of the algebra.

An element $x \neq \theta$ is said to be a *divisor of zero* if there is a $y \neq \theta$ such that either $xy = \theta$ or $yx = \theta$. It is clear that y is also a divisor of zero. This notion has been generalized by G. Šilov [1] who calls x a *generalized divisor of zero* if there is a sequence $\{y_n\} \subset \mathfrak{B}$ with $\| y_n \| = 1$ such that either $xy_n \to \theta$ or $y_n x \to \theta$. It is clear that such an element is singular.

An element $x \neq \theta$ is said to be *nilpotent* if some power of x equals θ. Such an element is clearly a divisor of zero. This notion has been generalized by I. Gelfand [4] (the term *quasi-nilpotent* is due to E. R. Lorch [3]); x is quasi-nilpotent if $\lim_{n\to\infty} \| x^n \|^{1/n} = 0$. An equivalent condition is that $e - \mu x$ shall be regular for every finite complex μ. Finally we have the *idempotent* elements: j is idempotent if $j^2 = j$. This condition is of course satisfied by the zero and the unit elements, but there may be other idempotents. Thus if $\mathfrak{B} = \mathfrak{E}(\mathfrak{X})$ and P is a projection operator, then we have $P^2 = P$. Any idempotent $j \neq e$ is singular.

References. Gelfand [4], Lorch [3], Nagumo [1], and Šilov [1].

2. Functions on Scalars to the Algebra

5.5. The calculus. As already remarked in the Orientation, the theory of functions on scalars to a Banach algebra differs from the corresponding theory of functions on scalars to a (B)-space only in the presence of ring multiplication. For this reason we restrict ourselves to pointing out some of the new features introduced by this fact. We also stick rather faithfully to the normed metric and all notions are based on convergence in this metric. We have thus for the most part only one form of each function theoretical concept to contend with.

In agreement with this convention, we say that *a function $x = x(\mathfrak{a})$ on a measurable set S in E_k to $\mathfrak{B}$ is measurable if it is the uniform limit almost everywhere of a sequence of countably-valued functions.* Theorem 3.2.3 holds for such functions and in addition *the product of two measurable functions is measurable.*

Continuity and differentiability are defined in the obvious manner in terms of the normed metric. The classical formulas of elementary differential calculus are largely valid for differentiable functions with values in $\mathfrak{B}$, but owing to the non-commutative character of multiplication, they may take unconventional

forms. Thus if $x(\xi)$ and $y(\xi)$ are differentiable functions in an interval (ξ_1, ξ_2), then $x(\xi)y(\xi)$ is also differentiable and

$$(5.5.1) \qquad [x(\xi)y(\xi)]' = x(\xi)y'(\xi) + x'(\xi)y(\xi),$$

where the order of the factors is essential. Similarly

$$(5.5.2) \qquad [x^{-1}(\xi)]' = -x^{-1}(\xi)x'(\xi)x^{-1}(\xi)$$

is valid if and only if $x(\xi)$ is a regular element of $\mathfrak{B}$ and $x(\xi)$ is differentiable. Differentiation of non-rational functions of a function requires considerations belonging to §5.4 and will be left to one side for the time being.

5.6. Differential equations. It is a priori fairly obvious that the classical existence theorems for differential equations may be extended to Banach algebras (and even to more general spaces as has been shown by A. Michal and his school). The following simple theorem is basic.

THEOREM 5.6.1. *Let $y = f(\xi, x)$ on $E_1 \times \mathfrak{B}$ to $\mathfrak{B}$ be defined and continuous in each variable separately for $|\xi - \xi_0| \leqq \alpha$, $||x - x_0|| \leqq \beta$ and satisfy*

$$||f(\xi, x)|| \leqq \mu, \quad ||f(\xi, x_1) - f(\xi, x_2)|| \leqq \gamma ||x_1 - x_2||$$

for ξ, x, x_1, x_2 in the regions indicated. Here α, β, γ, μ are fixed positive numbers and $\alpha\mu \leqq \beta$. Then there is one and only one solution $x = x(\xi)$ of the differential equation

$$\frac{dx}{d\xi} = f(\xi, x)$$

in $|\xi - \xi_0| \leqq \alpha$ such that $x(\xi_0) = x_0$.

PROOF. The classical method of successive approximations can be used and leads to a sequence of functions $x_n(\xi)$ defined for $|\xi - \xi_0| \leqq \alpha$ by

$$x_0(\xi) = x_0, \qquad x_n(\xi) = x_0 + \int_{\xi_0}^{\xi} f(\tau, x_{n-1}(\tau))\, d\tau,$$

the integrals being taken in the sense of Bochner (actually Riemann-Graves would suffice). The classical proof may be followed step by step, replacing absolute values by norms throughout. One proves successively that the approximations are continuous, differentiable functions of ξ in $|\xi - \xi_0| \leqq \alpha$ and that they form a Cauchy sequence. If then $\lim_{n\to\infty} x_n(\xi) = x(\xi)$, we have

$$||f(\xi, x(\xi)) - f(\xi, x_n(\xi))|| \leqq \gamma ||x(\xi) - x_n(\xi)|| \to 0.$$

Hence

$$x(\xi) = x_0 + \int_{\xi_0}^{\xi} f(\tau, x(\tau))\, d\tau$$

and

$$x(\xi_0) = x_0, \; x'(\xi) = f(\xi, x(\xi)).$$

The uniqueness is proved in the customary manner.

The special case

$$\frac{dx}{d\xi} = ax, \qquad x(0) = e, \qquad a \in \mathfrak{B}, \tag{5.6.1}$$

is of interest to the theory of semi-groups. Here the method of successive approximations leads to the unique solution

$$e + \sum_{n=1}^{\infty} \frac{\xi^n a^n}{n!} \equiv \exp \xi(a). \tag{5.6.2}$$

We take this series as the definition of the *exponential function* exp (ξa). It obviously reduces to the classical exponential function when $\mathfrak{B}$ is the algebra of complex numbers.

5.7. Complex function theory. The methods of §3.2 apply to holomorphic functions on a domain D of the complex plane to a Banach algebra $\mathfrak{B}$. In particular, a power series in $\zeta - \zeta_0$ with coefficients in $\mathfrak{B}$

$$\sum_{n=0}^{\infty} a_n(\zeta - \zeta_0)^n \equiv x(\zeta)$$

defines a holomorphic function on $| \zeta - \zeta_0 | < \rho$ to $\mathfrak{B}$ where

$$1/\rho = \limsup_{n \to \infty} \| a_n \|^{1/n}. \tag{5.7.1}$$

Incidentally, this Cauchy-Hadamard formula for the radius of convergence is also valid for power series with coefficients in a (B)-space. The series diverges outside the circle.

In the case of a Banach algebra, however, we may multiply absolutely convergent power series and the product is an absolutely convergent power series whose coefficients are given by the Cauchy formula so that

$$\sum_{n=0}^{\infty} a_n \zeta^n \sum_{n=0}^{\infty} b_n \zeta^n = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} a_k b_{n-k} \right] \zeta^n.$$

The order of the factors is of course essential. Further, if the constant term a_0 of $x(\zeta)$ is a regular element of $\mathfrak{B}$ so that a_0^{-1} exists, then $x(\zeta)$ has also a holomorphic inverse for sufficiently small values of $| \zeta - \zeta_0 |$. This follows from Theorem 5.2.3 and formula (5.5.2).

The formula

$$\exp(\zeta a) = e + \sum_{n=1}^{\infty} \frac{\zeta^n a^n}{n!}, \qquad a \in \mathfrak{B}, \tag{5.7.2}$$

clearly defines an entire function of ζ and, in agreement with (5.6.1), we have

$$\frac{d}{d\zeta} \exp(\zeta a) = a \exp(\zeta a) = \exp(\zeta a)\, a. \tag{5.7.3}$$

A simple calculation shows that if a_1 and a_2 commute then

$$\exp(\zeta_1 a_1)\exp(\zeta_2 a_2) = \exp(\zeta_1 a_1 + \zeta_2 a_2) \tag{5.7.4}$$

and, in particular,

$$\exp(\zeta a)\exp(-\zeta a) = e \tag{5.7.5}$$

for all ζ. Hence $\exp(\zeta a)$ has an inverse for all ζ and a. These formulas serve as further justification for the name exponential function assigned to the series in (5.7.2).

In conclusion let us remark that in dealing with analytic functions to a Banach algebra the analyst has to be prepared for functions with somewhat unconventional behavior; the similarity with the classical case is not merely suggestive but also deceptive. Thus, for instance, if q is a quasi-nilpotent element, the function $\log(e - q\zeta)$, defined by the obvious power series, is an entire function of ζ. Similarly the function $(e - a\zeta)^{-1}$ is ordinarily not a rational function of ζ. We shall make a thorough study of this function or rather of an equivalent function in the next paragraph.

References. Kerner [1], Nagumo [1].

3. Spectral Theory

5.8. The resolvent. We shall consider the inverse of $\lambda e - a$ as a function of λ for a fixed $a \in \mathfrak{B}$. The resulting theory is closely related to the discussion in section 2.14, where we considered the resolvent of a linear transformation T on a (B)-space to itself, and has direct and important applications to the theory of $R(\lambda; T)$. Our present point of view is somewhat different, however, inasmuch as we are interested only in what happens in the given space and not in any underlying space. Thus the classification of the spectrum into point spectrum, continuous spectrum, and residual spectrum has no longer any sense and we are merely concerned with the resolvent set and its complement the spectrum.

If we so desire, however, we may consider the element a as an operator defining the transformation $y = ax$ on $\mathfrak{B}$ to itself and to this operator we can apply the discussion in section 2.14. Thus we have to distinguish between the spectral properties of a as an element of $\mathfrak{B}$ and as an operator on $\mathfrak{B}$ to itself. It is properties of the first kind which concern us here.

Definition 5.8.1. *According as $\lambda e - a$ is regular or singular in $\mathfrak{B}$, we say that λ belongs to the resolvent set $\rho(a)$ or the spectrum $\sigma(a)$ of a. For λ in $\rho(a)$ the inverse of $\lambda e - a$ exists; it is denoted by $R(\lambda; a)$ and called the resolvent of a.*

We have from the definition

$$(5.8.1) \qquad R(\lambda;\, a)(\lambda e - a) = (\lambda e - a)R(\lambda;\, a) = e,\ \lambda \in \rho(a).$$

THEOREM 5.8.1. *The resolvent set is open. In each of its components $R(\lambda; a)$ is a holomorphic function of λ.*

PROOF. Suppose that $\lambda_0 \in \rho(a)$ and substitute $x = \lambda e - a$, $x_0 = \lambda_0 e - a$ in formula (5.2.2). Since $x_0 - x = (\lambda_0 - \lambda)e$, the formal result is

$$(5.8.2) \qquad R(\lambda;\, a) = R(\lambda_0\,;\, a)\left\{e + \sum_{n=1}^{\infty} (\lambda_0 - \lambda)^n [R(\lambda_0\,;\, a)]^n\right\}.$$

This series is absolutely convergent at least when

$$|\lambda - \lambda_0| < \| R(\lambda_0\,;\, a) \|^{-1}$$

and within this circle it defines a holomorphic function of λ. Multiplication by $\lambda e - a = (\lambda - \lambda_0)e + (\lambda_0 e - a)$ on left or right gives e, so the series actually represents the resolvent. This shows that a circular neighborhood of λ_0 also belongs to $\rho(a)$ and $R(\lambda; a)$ is holomorphic in this neighborhood. This proves the theorem.

As an open set, $\rho(a)$ is the union of a finite or countably infinite number of disjoint connected open sets, the *components* of $\rho(a)$. We shall see in a moment that *there is at least one component, $\rho_1(a)$, known as the principal component and containing the point at infinity.* But ordinarily there are infinitely many components. *Thus, $R(\lambda; a)$ is in general not an analytic function in the sense of Weierstrass, but rather an analytic expression defining distinct analytic functions in the distinct components.* We shall express this fact briefly by saying that *$R(\lambda; a)$ is locally analytic* or, since $R(\lambda; a)$ is always single-valued, *locally holomorphic.*

In studying the principal component we shall need the following

LEMMA 5.8.1. *Let $\{\alpha_n\}$ be a sequence of real numbers such that $\alpha_{m+n} \leqq \alpha_m + \alpha_n$ for all m and n. Then $\alpha = \lim_{n\to\infty} (\alpha_n/n)$ exists and $-\infty \leqq \alpha < +\infty$.*

For the simple proof we refer to G. Pólya and G. Szegö [1, p. 171, problem 98]. Actually $\alpha = \inf (\alpha_n/n)$.. The sequence $\{\alpha_n\}$ is subadditive in the subscript and the result is closely related to Theorem 6.6.1.

THEOREM 5.8.2. *We have*

$$R(\lambda;\, a) = e\lambda^{-1} + \sum_{n=1}^{\infty} a^n \lambda^{-n-1}$$

for $|\lambda| > \gamma$ where

$$\gamma = \lim_{n\to\infty} \| a^n \|^{1/n}.$$

PROOF. Putting

$$\alpha_n = \log \| a^n \|,$$

it is a simple matter to verify that $\alpha_{m+n} \leqq \alpha_m + \alpha_n$. Hence

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = \lim_{n\to\infty} \log \| a^n \|^{1/n} = \alpha$$

exists. We put $e^\alpha = \gamma$. That the series converges absolutely for $|\lambda| > \gamma$ follows from formula (5.7.1). Multiplication of the series by $(\lambda e - a)$ on either side gives e. Hence the series represents $R(\lambda; a)$ for $|\lambda| > \gamma$.

It follows that the exterior of the circle $|\lambda| = \gamma$ belongs to $\rho(a)$ and by definition it belongs to the principal component of $\rho(a)$.

THEOREM 5.8.3. *The spectrum of a is a closed non-vacuous point set.*

PROOF. The spectrum, being the complement of the open resolvent set, is necessarily closed. If $\sigma(a)$ were vacuous, then $\rho(a)$ would be the whole complex plane and $R(\lambda; a)$ would be an entire function which is also holomorphic at infinity. By the extended Liouville Theorem 3.12.2, $R(\lambda; a)$ must be a constant and, since $R(\lambda; a) \to \theta$ when $\lambda \to \infty$, $R(\lambda; a) \equiv \theta$ which clearly contradicts formula (5.8.1).

Any closed bounded point set in the λ-plane may be the spectrum of an element in a suitably chosen Banach algebra. Thus if $x(\xi)$ is a bounded complex-valued function defined on (0, 1) and $\| x(\cdot) \| = \sup | x(\xi) |$, then the set of all such functions with the arithmetical operations defined in the ordinary manner is a Banach algebra. Here the spectrum of $x(\cdot)$ is clearly the closure of its range which may be a perfectly arbitrary closed bounded point set.

That $R(\lambda; a)$ is locally analytic was proved above with the aid of the series (5.8.2). A more direct proof could be had from formula (5.5.2) and the remark in section 5.7 according to which the inverse of a holomorphic function is holomorphic wherever it exists. Applied to the holomorphic function $\lambda e - a$, it shows that *$R(\lambda; a)$ is holomorphic wherever it exists.*

For future reference we note the formula for the derivatives of $R(\lambda; a)$ which may be read off from (5.8.2):

$$R^{(k)}(\lambda; a) = (-1)^k k! \, [R(\lambda; a)]^{k+1}. \tag{5.8.3}$$

5.9. The resolvent equation. A basic property of the resolvent is contained in the resolvent equation:

THEOREM 5.9.1. *If λ and μ belong to $\rho(a)$, then*

$$R(\lambda; a) - R(\mu; a) = -(\lambda - \mu) R(\lambda; a) R(\mu; a). \tag{5.9.1}$$

REMARK. We shall later encounter a *second resolvent equation.* (5.9.1) will then be called the *first resolvent equation.*

PROOF. We have

$$\begin{aligned} R(\lambda;\, a) &= R(\lambda;\, a)(\mu e - a)R(\mu;\, a) \\ &= R(\lambda;\, a)[(\mu - \lambda)e + (\lambda e - a)]R(\mu;\, a) \\ &= (\mu - \lambda)R(\lambda;\, a)R(\mu;\, a) + R(\mu;\, a). \end{aligned}$$

Incidentally, it follows from (5.9.1) that $R(\lambda;\, a)$ and $R(\mu;\, a)$ commute. It should be observed that it is not necessary for λ and μ to belong to the same component of $\rho(a)$. We note also that the resolvent equation is satisfied by all resolvents $R(\lambda;\, a)$. The value of a determines the domain in the complex plane where the equation is satisfied.

The functional equation

$$R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu), \qquad R(\lambda) \in \mathfrak{B} \tag{5.9.2}$$

is of great importance in analysis. The equation imposes quite severe restrictions on the solutions which have to be analytic in λ. All resolvents are solutions, but we shall see that there are other solutions as well.

THEOREM 5.9.2 *Let $R(\lambda)$ be a single-valued function on a domain D of the complex plane to a Banach algebra $\mathfrak{B}$, satisfying* (5.9.2) *for all values of λ and μ in D. Then $R(\lambda)$ is holomorphic in D. A necessary and sufficient condition that $R(\lambda)$ be the resolvent of some element of $\mathfrak{B}$ is that $R(\lambda)$ be regular for at least one value of λ in D.*

PROOF. The equation (5.9.2) shows that $R(\lambda)$ and $R(\mu)$ commute. Let $\lambda_0 \in D$ and replace μ by λ_0 in the equation, obtaining

$$R(\lambda)[e + (\lambda - \lambda_0)R(\lambda_0)] = R(\lambda_0).$$

By Theorem 5.2.1 the second factor on the left has an inverse at least for $|\lambda - \lambda_0|\, \|R(\lambda_0)\| < 1$ and multiplying both sides by the inverse gives

$$R(\lambda) = R(\lambda_0)\left\{e + \sum_{n=1}^{\infty} (\lambda_0 - \lambda)^n [R(\lambda_0)]^n\right\}.$$

Hence $R(\lambda)$ is holomorphic in a neighborhood of $\lambda = \lambda_0$ and, λ_0 being arbitrary, this means that $R(\lambda)$ is holomorphic in D.

If $R(\lambda)$ is the resolvent of an element a of $\mathfrak{B}$ for $\lambda \in D$, then $R(\lambda)$ is regular for every $\lambda \in \rho(a)$ and in particular in D. It follows that the stated condition is necessary. But it is also sufficient. Suppose that $[R(\lambda_0)]^{-1}$ exists and put $a = \lambda_0 e - [R(\lambda_0)]^{-1}$. Then $\lambda e - a = (\lambda - \lambda_0)e + [R(\lambda_0)]^{-1}$ and by (5.9.2)

$$\begin{aligned} R(\lambda)\{(\lambda - \lambda_0)e + [R(\lambda_0)]^{-1}\} &= (\lambda - \lambda_0)R(\lambda) + R(\lambda)[R(\lambda_0)]^{-1} \\ &= (\lambda - \lambda_0)R(\lambda) + e - (\lambda - \lambda_0)R(\lambda) = e. \end{aligned}$$

Thus $R(\lambda)$ is the resolvent of a. It is easily seen that a is actually independent of λ_0.

Corollary. *Every solution of the resolvent equation which is defined and single-valued in some neighborhood of $\lambda = \lambda_0$ is of the form*

$$R(\lambda) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n x^{n+1}, \tag{5.9.3}$$

where x is an element of $\mathfrak{B}$. Conversely, every such series satisfies the resolvent equation in its circle of convergence. It is a resolvent if and only if x is regular.

If x is singlar we call the function $R(\lambda)$ defined by (5.9.3) a *pseudo-resolvent*. Using Theorem 5.8.2 we may express $R(\lambda)$ in terms of a resolvent by the formula

$$R(\lambda) = (\lambda_0 - \lambda)^{-1} x R[(\lambda_0 - \lambda)^{-1}; x].$$

In particular, *if $x = q$ is quasi-nilpotent, then $R(\lambda)$ is an entire function of λ which reduces to a polynomial if q is actually nilpotent.* Conversely, *if a solution $R(\lambda)$ of the resolvent equation is an entire function of λ, then $R(\lambda)$ is quasi-nilpotent in $\mathfrak{B}$ for every λ.*

The resolvent equation admits of several simple transformations which leave it invariant and which consequently enable us to construct new solutions. The following results are worthy of notice; the verifications are left to the reader.

(1) *If $R(\lambda)$ is a solution, so is $R(\lambda + \alpha)$ for any fixed α.*

(2) *If j is an idempotent in $\mathfrak{B}$ and j commutes with $R(\lambda)$, then $jR(\lambda)$ is also a solution.*

(3) *If $R(\lambda)$ is a solution so is*

$$\frac{e}{\lambda} + \frac{1}{\lambda^2} R\left(-\frac{1}{\lambda}\right).$$

In connection with (2) the following remark should be made. If $R(\lambda)$ is a resolvent, $R(\lambda) = R(\lambda; a)$, then $jR(\lambda)$ is a *quasi-resolvent* in the sense that

$$(j\lambda - ja) jR(\lambda; a) = j, \tag{5.9.4}$$

that is, $jR(\lambda; a)$ may be regarded as the resolvent of ja in the subalgebra $j\mathfrak{B}j$ in which j plays the role of unit element.

We come finally to solutions which are holomorphic at infinity.

Theorem 5.9.3. *Every solution of the resolvent equation which is bounded outside a large circle is of the form*

$$R(\lambda) = z + jR(\lambda; x) \tag{5.9.5}$$

where $z^2 = \theta$, $j^2 = j$, $zj = jz = \theta$, $xj = jx = x$, that is, $x \in j\mathfrak{B}j$. Such a solution is a quasi-resolvent if $z = \theta$, a resolvent if $z = \theta$, $j = e$.

Proof. Since $R(\lambda)$ is holomorphic at every finite point where it is defined and bounded at infinity, it must be holomorphic also at infinity. Substituting $R(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^{-n}$ in (5.9.2), dividing by $(\lambda - \mu)$, and comparing coefficients of

like powers, we see first that all coefficients commute and obtain the following system of equations:

$$c_0^2 = \theta, \qquad c_0c_k = \theta \text{ for all } k,$$
$$c_1^2 = c_1, \qquad c_n = c_{n-j}c_{j+1}, \qquad n > 1, j = 0, 1, \cdots, n-1.$$

The last set of equations shows that $c_n = (c_2)^{n-1}$, $n > 1$. Putting $c_0 = z$, $c_1 = j$, $c_2 = x$, we obtain the expression stated in the theorem and the relations between z, j and x are read off from the equations. The remaining assertions then become obvious.

5.10. The Laurent resolution of the spectrum. Consideration of the Laurent expansion of a solution $R(\lambda)$ of the resolvent equation leads to important conclusions concerning the structure of $R(\lambda)$. Since the resolvent equation is unchanged under translations, it is no restriction in assuming that the center of the annulus of holomorphism of $R(\lambda)$ is at $\lambda = 0$.

THEOREM 5.10.1. *If $R(\lambda)$ is a solution of the resolvent equation for $0 \leqq \gamma_1 < |\lambda| < \gamma_2 \leqq \infty$, then*

$$R(\lambda) = R^-(\lambda) + R^+(\lambda),$$
$$R^-(\lambda)R^+(\lambda) = R^+(\lambda)R^-(\lambda) = \theta, \tag{5.10.1}$$

where

(i) *$R^-(\lambda)$ and $R^+(\lambda)$ are also solutions of the resolvent equation;*

(ii) *$R^-(\lambda)$ is holomorphic for $|\lambda| > \gamma_1$ and is of the form*

$$R^-(\lambda) = jR(\lambda; a^-),$$

j being an idempotent and $a^- \in j\mathfrak{B}j$, if $\gamma_1 = 0$, a^- is quasi-nilpotent;

(iii) *$R^+(\lambda)$ is holomorphic for $|\lambda| < \gamma_2$ and*

$$R^+(\lambda) = \sum_{n=0}^{\infty} (-\lambda)^n a_0^{n+1},$$

a_0 belonging to the subalgebra $(e - j)\mathfrak{B}(e - j)$.

In particular, if $R(\lambda)$ is the resolvent of the element a, then

$$a = a^+ + a^-,$$

where

$$a^- = ja, \ a^+ = (e - j)a, \ -a_0a^+ = e - j,$$

and

$$[(e - j)\lambda - a^+]R^+(\lambda) = (e - j),$$
$$(j\lambda - a^-)R^-(\lambda) = j.$$

REMARK. This theorem is suggested by results of M. Nagumo [1] who discussed the case in which $R(\lambda) = R(\lambda; a)$ and the center of the annulus is an isolated singularity on the boundary of the principal component of $\rho(a)$. Nagumo, however, stated that the functions $R^+(\lambda)$ and $R^-(\lambda)$ are resolvents; properly speaking they are merely quasi-resolvents as will be seen.

PROOF. Substituting a Laurent series in the resolvent equation, we obtain

$$\sum_{-\infty}^{\infty} c_n \frac{\lambda^n - \mu^n}{\lambda - \mu} = -\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} c_k c_m \lambda^k \mu^m.$$

It is easily seen that all the elements c_n commute with each other. Also on the left-hand side of the above equation the coefficient of c_n is

$$\begin{aligned} &\lambda^{n-1} + \lambda^{n-2}\mu + \cdots + \mu^{n-1}, && \text{if } n \geqq 1; \\ &0, && \text{if } n = 0; \\ &-(\lambda^n\mu^{-1} + \lambda^{n+1}\mu^{-2} + \cdots + \lambda^{-1}\mu^n), && \text{if } n < 0. \end{aligned}$$

Thus all terms involving λ^n and μ^n, n being negative, are missing as well as those involving mixed products $\lambda^k\mu^m$ in which the exponents have opposite signs. This implies that every element c_k with $k \geqq 0$ is orthogonal to every c_m with $m \leqq -1$. Then defining

$$R^+(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n, \qquad R^-(\lambda) = \sum_{n=-\infty}^{-1} c_n \lambda^n,$$

one sees that relations (5.10.1) hold. Substituting this expression for $R(\lambda)$ in the resolvent equation and using the orthogonality property, it may be seen from the power series involved that both $R^+(\lambda)$ and $R^-(\lambda)$ will be solutions of the resolvent equation.

$R^-(\lambda)$ is holomorphic at infinity and vanishes there. Hence from Theorem 5.9.3, $R^-(\lambda) = jR(\lambda; a^-)$ where $a^- \in j\mathfrak{B}j$. The series defining $R(\lambda; a^-)$ converges for $|\lambda| > \gamma_1$ and if $\gamma_1 = 0$ this implies that a^- is a quasi-nilpotent element. The idempotent j cannot be θ unless $R^-(\lambda) \equiv \theta$ and it will be seen in a moment that $j \neq e$ unless $R^+(\lambda) \equiv \theta$.

Since $R^+(\lambda)$ is holomorphic at the origin the corollary of Theorem 5.9.2 shows that $R^+(\lambda) = \sum_{n=0}^{\infty} (-\lambda)^n a_0^{n+1}$. Since $a_0 = c_0$ and $j = c_{-1}$, these two elements commute and are orthogonal to each other. It follows that a_0 belongs to the subalgebra $(e - j)\mathfrak{B}(e - j)$. The assumption $j = e$ would clearly imply $a_0 = \theta$ and $R^+(\lambda) \equiv \theta$.

$R^+(\lambda)$ being orthogonal to j, we conclude that

$$R^-(\lambda) = jR(\lambda) = R(\lambda)j, \qquad R^+(\lambda) = (e - j)R(\lambda) = R(\lambda)(e - j),$$

and

$$R^-(\lambda)a_0 = a_0R^-(\lambda) = \theta, \qquad R^+(\lambda)a^- = a^-R^+(\lambda) = \theta.$$

If $R(\lambda)$ is the resolvent of a,

$$(\lambda e - a)R(\lambda) = R(\lambda)(\lambda e - a) = e;$$

upon multiplication by the idempotents j and $e - j$ this identity yields

$$(j\lambda - ja)R^-(\lambda) = R^-(\lambda)(j\lambda - ja) = j,$$

$$[(e-j)\lambda - (e-j)a]R^+(\lambda) = R^+(\lambda)[(e-j)\lambda - (e-j)a] = (e-j).$$

On the other hand, from the definition of $R^-(\lambda)$ it follows that

$$(j\lambda - a^-)R^-(\lambda) = R^-(\lambda)(j\lambda - a^-) = j.$$

Since a^- and ja are both in $j\mathfrak{B}j$, the fact that they have the same quasi-resolvent in this subalgebra implies that they are equal, $a^- = ja$. Putting $a^+ = a - a^- = (e - j)a$, it is seen that

$$-a^+R^+(0) = -a^+a_0 = e - j$$

and so a^+ is the quasi-inverse of $-a_0$ in the subalgebra $(e - j)\mathfrak{B}(e - j)$. The remaining parts of the theorem are at once verified.

Corollary. *If θ is the only quasi-nilpotent element in $\mathfrak{B}$, then any isolated singularity of $R(\lambda)$ is necessarily a simple pole.*

Indeed, if the isolated singularity is at the origin as we may assume, the principal part of $R(\lambda)$ there is $R^-(\lambda) = jR(\lambda; a^-)$ where a^- has to be quasi-nilpotent and hence equal to θ. $R^-(\lambda)$ then reduces to its first term j/λ and $R(\lambda)$ has a simple pole. In the terminology of Definition 22.13.2, $\mathfrak{B}$ is without a radical.

For the case in which $R(\lambda) = R(\lambda; a)$, Theorem 5.10.1 asserts that *to a particular resolution of the spectrum of a into two complementary sets corresponds a resolution of a and of $R(\lambda; a)$ into orthogonal components.* This is merely a special case of a more general resolution theorem which will be proved in the next section.

5.11. Spectral sets. The basic notion here is that of a *spectral set* introduced by N. Dunford. We shall also need the notion of *an oriented envelope of a spectral set.*

Definition 5.11.1. *A non-void set σ is called a spectral set of a if σ is a subset of $\sigma(a)$ and σ is both open and closed in $\sigma(a)$. $\Gamma(\sigma)$ is called an oriented envelope of σ if*

(i) *$\Gamma(\sigma)$ is the union of a finite number of closed simple rectifiable curves having no points in common;*

(ii) *$\Gamma(\sigma)$ lies in the resolvent set $\rho(a)$ of a;*

(iii) *$\Gamma(\sigma)$ bounds an open set $\Delta(\sigma)$ containing σ;*

(iv) *$\sigma' = \sigma(a) - \sigma$ belongs to the complement of $\Gamma(\sigma) \cup \Delta(\sigma)$:*

(v) $\max_{\lambda\in\Gamma(\sigma)} d(\lambda, \sigma) < \frac{1}{3}d(\sigma, \sigma')$;
(vi) *in the positive orientation of* $\Gamma(\sigma)$, *the set* $\Delta(\sigma)$ *lies to the left of* $\Gamma(\sigma)$.

Condition (v) ensures that the closure of a $\Delta(\sigma)$ has no points in common with the closure of a $\Delta(\sigma')$. The construction of $\Gamma(\sigma)$ can be based on a simple Heine-Borel argument which is left to the reader.

To each resolution of $\sigma(a)$ into disjoint spectral sets there is a corresponding resolution of a and $R(\lambda; a)$ given in the following

THEOREM 5.11.1. *Let* $\sigma(a) = \bigcup_1^k \sigma_\alpha$ *where each* σ_α *is a spectral set of* a *and* $\sigma_\alpha \cap \sigma_\beta = \emptyset$ *when* $\alpha \neq \beta$. *Define*

$$j_\alpha = \frac{1}{2\pi i}\int_{\Gamma_\alpha} R(\zeta; a)\, d\zeta, \tag{5.11.1}$$

where Γ_α *is an oriented envelope of* σ_α. *Then*

$$\sum_1^k j_\alpha = e, \qquad j_\alpha^2 = j_\alpha, \qquad j_\alpha j_\beta = \theta, \qquad \alpha \neq \beta, \qquad j_\alpha \neq \theta, e.$$

Setting

$$a_\alpha = j_\alpha a, \qquad R_\alpha(\lambda; a) = j_\alpha R(\lambda; a),$$

then

$$R(\lambda; a) = \sum_1^k R_\alpha(\lambda; a), \qquad R_\alpha(\lambda; a) R_\beta(\lambda; a) = \theta, \qquad \alpha \neq \beta;$$

$$a = \sum_1^k a_\alpha, \qquad a_\alpha a_\beta = \theta, \qquad \alpha \neq \beta;$$

$$R_\alpha(\lambda; a) a_\beta = a_\beta R_\alpha(\lambda; a) = \theta, \qquad \alpha \neq \beta;$$

$$(j_\alpha \lambda - a_\alpha) R_\alpha(\lambda; a) = R_\alpha(\lambda; a)(j_\alpha \lambda - a_\alpha) = j_\alpha .$$

Furthermore, the spectrum of a_α *is* σ_α *in addition to* $\lambda = 0$ *and* $R_\alpha(\lambda; a)$ *is holomorphic in each domain of the complex plane which does not contain any points of* σ_α. *Finally*

$$R_\alpha(\lambda; a) = j_\alpha \lambda^{-1} + \sum_1^\infty a_\alpha^n \lambda^{-n-1}$$

for $|\lambda| > \gamma_\alpha = \lim_{n\to\infty} \|a_\alpha^n\|^{1/n}$.

PROOF. It is first observed that if Γ is a circle $|\lambda| = \gamma > \|a\|$, then

$$\sum_1^k j_\alpha = \sum_1^k \frac{1}{2\pi i}\int_{\Gamma_\alpha} R(\zeta; a)\, d\zeta = \frac{1}{2\pi i}\int_\Gamma R(\zeta; a)\, d\zeta = e.$$

By the resolvent equation

$$j_\alpha j_\beta = \frac{1}{(2\pi i)^2} \int_{\Gamma_\alpha} \int_{\Gamma_\beta} R(\lambda;\, a) R(\mu;\, a)\, d\lambda\, d\mu$$

$$= \frac{1}{2\pi i} \int_{\Gamma_\alpha} R(\lambda;\, a)\, d\lambda\, \frac{1}{2\pi i} \int_{\Gamma_\beta} \frac{d\mu}{\mu - \lambda}$$

$$- \frac{1}{2\pi i} \int_{\Gamma_\beta} R(\mu;\, a)\, d\mu\, \frac{1}{2\pi i} \int_{\Gamma_\alpha} \frac{d\lambda}{\mu - \lambda}.$$

If $\alpha = \beta$ replace Γ_β by Γ'_α, another oriented envelope of σ_α containing Γ_α in its interior. In this case the two integrals involving $(\mu - \lambda)^{-1}$ are equal to $2\pi i$ and 0 respectively and so $j_\alpha^2 = j_\alpha$. If $\alpha \neq \beta$ each integral is zero and $j_\alpha j_\beta = \theta$.

We note next that j_α and a commute by formula (5.11.1) since a commutes with $R(\lambda;\, a)$. Furthermore j_α and $R(\lambda;\, a)$ commute by the same formula since the resolvents $R(\lambda;\, a)$ and $R(\mu;\, a)$ commute. The orthogonality properties of a_α and $R_\alpha(\lambda;\, a)$ are immediate consequences of those of j_α. The quasi-resolvent property follows from multiplying $(\lambda e - a)R(\lambda;\, a) = e$ by $j_\alpha^2 = j_\alpha$. The given representation of $R_\alpha(\lambda;\, a)$ for $|\lambda| > \gamma_\alpha$ then follows immediately.

From the resolvent equation we obtain, upon dividing by $(\lambda - \mu)$ and integrating with respect to μ around the contour Γ_α, a result which may be written

$$R_\alpha(\lambda;\, a) = -\frac{1}{2\pi i} \int_{\Gamma_\alpha} \frac{R(\mu;\, a)}{\mu - \lambda}\, d\mu + R(\lambda;\, a)\, \frac{1}{2\pi i} \int_{\Gamma_\alpha} \frac{d\mu}{\mu - \lambda}.$$

For λ outside of $\bar{\Delta}(\sigma_\alpha)$, the first integral, being of the Cauchy type, is a holomorphic function of λ, while the second integral is zero. Replacing Γ_α by Γ'_α as above, the first integral is holomorphic for λ in $\Delta'(\sigma_\alpha)$, while the second term becomes $R(\lambda;\, a)$. Hence $R_\alpha(\lambda;\, a)$ is holomorphic outside any oriented envelope of σ_α and inside Γ_α it has exactly the same singularities as $R(\lambda;\, a)$, that is, the set σ_α. This shows also that $j_\alpha \neq \theta,\, e$.

Finally we consider the spectrum of a_α. For $|\lambda| > \gamma_\alpha$

$$R(\lambda;\, a_\alpha) = e\lambda^{-1} + \sum_1^\infty a_\alpha^n \lambda^{-n-1} = (e - j_\alpha)\lambda^{-1} + R_\alpha(\lambda;\, a).$$

The singularities of the last member are clearly 0 and σ_α. This completes the proof of the theorem.

The possibilities of extending this resolution theory to infinitely many components will not be considered here.

The following theorem follows from the results of Theorems 5.10.1 and 5.11.1.

THEOREM 5.11.2. *Let the resolvent $R(\lambda;\, a)$ be holomorphic except for isolated singular points $\lambda_1, \lambda_2, \cdots, \lambda_k$. Then there exist k idempotents $j_1, j_2, \cdots, j_k$ where*

$$j_\alpha = \frac{1}{2\pi i} \int_{\Gamma_\alpha} R(\zeta;\, a)\, d\zeta,$$

Γ_α being a small circle about λ_α, such that

$$R(\lambda; a) = \sum_1^k R_\alpha^-(\lambda), \qquad R_\alpha^-(\lambda)R_\beta^-(\lambda) = \theta, \qquad \alpha \neq \beta,$$

and

$$R_\alpha^-(\lambda) = j_\alpha R(\lambda; a) = j_\alpha(\lambda - \lambda_\alpha)^{-1} + \sum_1^\infty (a_\alpha^-)^n(\lambda - \lambda_\alpha)^{-n-1}.$$

Here $a_\alpha^- = j_\alpha(a - \lambda_\alpha e)$ is quasi-nilpotent (or nilpotent). Furthermore, $a_\alpha^- a_\beta^- = \theta$ and $a = \sum_1^k a_\alpha^- + \sum_1^k j_\alpha\lambda_\alpha$.

Thus, to a complete resolution of the spectrum of a corresponds a complete resolution of a and of the resolvent of a.

5.12. Some special resolvents. We shall list the properties of $R(\lambda; a)$ for some particular choices of a.

(1) $a = q$ is *quasi-nilpotent*. Then

$$R(\lambda; q) = e\lambda^{-1} + \sum_{n=1}^\infty q^n \lambda^{-n-1}$$

which is an entire function of $1/\lambda$. The spectrum reduces to one point, $\lambda = 0$. If q is actually nilpotent, the series reduces to a polynomial in $1/\lambda$.

(2) $a = j$ is an *idempotent*. Then the resolvent series gives

$$R(\lambda; j) = (e - j)\lambda^{-1} + j(\lambda - 1)^{-1}.$$

In this case the spectrum consists of two points, $\lambda = 0$ and 1, each of which is a simple pole of the resolvent.

(3) $a = \alpha e$ is a *scalar multiple of the unit element*. It is easily verified that $R(\lambda; a)$ has the same property and

$$R(\lambda; \alpha e) = (\lambda - \alpha)^{-1}e$$

with a simple pole at $\lambda = \alpha$.

In connection with (2) the following theorem due to E. R. Lorch is of interest:

THEOREM 5.12.1. *A necessary and sufficient condition that a Banach algebra $\mathfrak{B}$ shall contain an idempotent different from θ and e is that there is at least one element of $\mathfrak{B}$ whose spectrum is not connected.*

PROOF. By virtue of (2) above, the condition is clearly necessary and by Theorem 5.11.1 it is also sufficient.

5.13. Permanent spectral singularities. It may be recalled that the regular or singular character of an element depends on the algebra in which it is embedded. If $a \in \mathfrak{B}_1 \subset \mathfrak{B}_2$ and a is regular in $\mathfrak{B}_1$, then it is also regular in $\mathfrak{B}_2$ provided that the extension preserves the arithmetical operations and the unit element. However, an element may be regular in $\mathfrak{B}_2$ but not in $\mathfrak{B}_1$; in other words, an element a as a rule has two different spectra in $\mathfrak{B}_1$ and $\mathfrak{B}_2$, $\sigma(a; \mathfrak{B}_1)$ and $\sigma(a; \mathfrak{B}_2)$ respectively with $\sigma(a; \mathfrak{B}_1) \supset \sigma(a; \mathfrak{B}_2)$. The following

definition is due to E. R. Lorch; Theorem 5.13.1 is due to C. E. Rickart (cf. S. Bochner and B. S. Phillips), the first half of Theorem 5.13.2 is due to E. R. Lorch and the second half to G. Šilov (for the case of accessible singularities).

Definition 5.13.1. *A point $\lambda \in \sigma(a; \mathfrak{B}_1)$ is a removable spectral singularity of a in $\mathfrak{B}_1$ if $\lambda \notin \sigma(a; \mathfrak{B}_2)$ for a suitable choice of $\mathfrak{B}_2 \supset \mathfrak{B}_1$. It is a permanent spectral singularity of a if $\lambda \in \sigma(a; \mathfrak{B}_2)$ for every possible extension $\mathfrak{B}_2$ of $\mathfrak{B}_1$.*

Theorem 5.13.1. *If $a_n \to a_0$ and a_n^{-1} exists for each n, then either a_0^{-1} exists or a_0 is a generalized divisor of zero.*

Proof. Suppose a_0^{-1} does not exist. We prove that $\limsup_{n\to\infty} \| a_n^{-1} \| = \infty$. Suppose $\| a_n^{-1} \| \leqq M < \infty$, for all n. Then

$$\| a_m^{-1} - a_n^{-1} \| = \| a_m^{-1}(a_n - a_m)a_n^{-1} \| \leqq M^2 \| a_n - a_m \| \to 0,$$

that is, there exists a b_0 such that $a_n^{-1} \to b_0$. Moreover

$$a_0 b_0 = (a_0 - a_n)b_0 + a_n(b_0 - a_n^{-1}) + a_n a_n^{-1}.$$

Since $(a_0 - a_n)b_0 \to \theta$, $a_n(b_0 - a_n^{-1}) \to \theta$, and $a_n a_n^{-1} = e$ we have $a_0 b_0 = e$. Similarly $b_0 a_0 = e$. This is a contradiction; therefore $\limsup_{n\to\infty} \| a_n^{-1} \| = \infty$ and we may clearly assume that this holds with lim sup replaced by lim. Now define $b_n = a_n^{-1} / \| a_n^{-1} \|$ so that $\| b_n \| = 1$. We have

$$a_0 b_n = (a_0 - a_n)b_n + a_n b_n,$$

where the two summands on the right tend to θ when $n \to \infty$. Therefore $a_0 b_n \to \theta$; that is, a_0 is a generalized divisor of zero.

Corollary. *If $\mathfrak{G}$ is the maximal group in $\mathfrak{B}$, then the elements on the boundary of $\mathfrak{G}$ are generalized divisors of zero.*

Another consequence is:

Theorem 5.13.2. *All points belonging to the boundary of the spectrum of an element a in $\mathfrak{B}$ are permanent spectral singularities of a. If λ_0 is such a boundary point, then $\lambda_0 e - a$ is a generalized divisor of zero.*

Proof. The second half of the theorem follows directly from the preceding theorem and this shows that $\lambda_0 e - a$ is a singular element of $\mathfrak{B}$ as well as of any extension of $\mathfrak{B}$.

This theorem also has interesting consequences.

Corollary 1. *The components of $\rho(a; \mathfrak{B}_1)$ are also components of $\rho(a; \mathfrak{B}_2)$ if $\mathfrak{B}_1 \subset \mathfrak{B}_2$.*

Corollary 2. *The spectrum $\sigma(a; \mathfrak{B})$ remains invariant under* (i) *extension of $\mathfrak{B}$ if $\sigma(a; \mathfrak{B})$ is nowhere dense,* (ii) *contraction of $\mathfrak{B}$ if $\rho(a; \mathfrak{B})$ is connected.*

5.14. Resolvents of linear operators. We shall now take up the problem of fitting the theory of resolvents of linear operators, first broached in section 2.14, into the theory of resolvents of elements of a Banach algebra developed in the preceding sections.

We take $\mathfrak{B} = \mathfrak{E}(\mathfrak{X})$, that is, the Banach algebra of linear bounded transformations on a (B)-space $\mathfrak{X}$ to itself. Suppose first that $T \in \mathfrak{E}(\mathfrak{X})$ and let $R(\lambda; T)$ be its resolvent in the sense of Definition 2.14.1. Thus $R(\lambda; T) \in \mathfrak{E}(\mathfrak{X})$ for λ in $\rho(T)$ but not for any λ in $\sigma(T)$. It follows that $R(\lambda; T)$ is the inverse of $\lambda I - T$ in the algebra $\mathfrak{E}(\mathfrak{X})$ and the notions of resolvent set and spectrum derived from Definition 5.8.1 coincide with the corresponding notions of Definition 2.14.1.

From this it follows that all results derived in the preceding sections apply to $R(\lambda; T)$ as an element of $\mathfrak{E}(\mathfrak{X})$. Thus $\rho(T)$ is an open set, $\sigma(T)$ is closed and never vacuous, $R(\lambda; T)$ is holomorphic in each component of $\rho(T)$ and, in particular, in a neighborhood of the point at infinity. The resolvent equation is satisfied and the various series expansions based on the resolvent equation are valid. Further the discussion of the resolution of the spectrum applies. What we do not get, however, is the resolution of the spectrum $\sigma(T)$ into the three components $P\sigma(T)$, $C\sigma(T)$, and $R\sigma(T)$ since this involves conditions in the underlying space $\mathfrak{X}$. The following result, however, can be stated in terms of properties of $R(\lambda; T)$ in $\mathfrak{E}(\mathfrak{X})$ alone, though the proof requires the use of properties in $\mathfrak{X}$. See A. E. Taylor [7, p. 660].

THEOREM 5.14.1. *If λ_0 is a pole of $R(\lambda; T)$, it belongs to the point spectrum of T.*

PROOF. By Theorems 5.10.1 and 5.11.2 the canonical representation of $R(\lambda; T)$ near $\lambda = \lambda_0$ is

$$R(\lambda; T) = J \sum_{n=0}^{\infty} Q^n(\lambda - \lambda_0)^{-n-1} + R^+(\lambda; T),$$

where $J^2 = J$, $Q = J(T - \lambda_0 I) = (T - \lambda_0 I)J$. If λ_0 is a pole of order m, $Q^{m-1} \neq \Theta$ but $Q^m = \Theta$. We can then find an element $y \in \mathfrak{X}$ such that $x = JQ^{m-1}y \neq \theta$. Hence $(T - \lambda_0 I)x = Q^m y = \theta$. Thus x is a characteristic element corresponding to the characteristic value λ_0. It is worth noting that if x is any characteristic element then $Jx = x$. This follows from the identity

$$(\lambda - \lambda_0)R(\lambda; T)x = x,$$

valid for characteristic elements, by substituting the canonical representation and comparing coefficients.

On the other hand, if $\lambda = \lambda_0$ is an essential singularity of $R(\lambda; T)$, no assertion can be made concerning the spectral character of λ_0. In the following three examples, T is a quasi-nilpotent operator, $\lambda = 0$ is an essential singular point of $R(\lambda; T)$ and is the only point of $\sigma(T)$:

(i) $\mathfrak{X} = (l_2)$, T takes $(x_1, x_2, \cdots, x_n, \cdots)$ into $(x_2, x_3/2, \cdots, x_{n+1}/n, \cdots)$, point spectrum;

(ii) $\mathfrak{X} = C[0, 1]$, T takes $x(t)$ into $\int_0^t x(u)\, du$, residual spectrum;

(iii) $\mathfrak{X} = C_0[0, 1]$, subspace of $C[0, 1]$ where $x[0] = 0$, T as in (ii), continuous spectrum.

For the last two examples, cf. section 21.12 where, however, $\mathfrak{X} = L[0, 1]$.

The same ambiguity is encountered at non-isolated points of the spectrum. Even an interior point may belong to the point spectrum. The following example illustrates this possibility:

(iv) $\mathfrak{X} = C[0, \infty]$, T takes $x(t)$ into $x(t + h)$ where h is a fixed positive number. Here the point spectrum is the set $|\lambda| < 1$ plus $\lambda = 1$, the rest of the unit circle is the continuous spectrum. Cf. section 16.2.

The case in which T is a linear unbounded transformation is less direct. We assume that T is closed; its domain $\mathfrak{D}$ and range $\mathfrak{R}$ are in $\mathfrak{X}$. Then $T_\lambda = \lambda I - T$ will be closed with domain $\mathfrak{D}$ and range $\mathfrak{R}_\lambda$. In case T_λ^{-1} exists, it will also be a closed linear transformation by Theorem 2.13.10. For $\lambda \in \rho(T)$, $T_\lambda^{-1} \equiv R(\lambda; T)$ exists as a linear bounded transformation whose domain $\mathfrak{R}_\lambda$ is dense in $\mathfrak{X}$ and since $R(\lambda; T)$ is closed, we have $\mathfrak{R}_\lambda = \mathfrak{X}$. Hence $R(\lambda; T) \in \mathfrak{E}(\mathfrak{X})$ and

$$(\lambda I - T)R(\lambda; T)x = x \text{ in } \mathfrak{X}, \qquad R(\lambda; T)(\lambda I - T)x = x \text{ in } \mathfrak{D}.$$

If $\lambda, \mu \in \rho(T)$, then for all x

$$\begin{aligned} R(\lambda; T)x &= R(\lambda; T)(\mu I - T)R(\mu; T)x \\ &= R(\lambda; T)[(\mu - \lambda)I + (\lambda I - T)]R(\mu; T)x \\ &= -(\lambda - \mu)R(\lambda; T)R(\mu; T)x + R(\mu; T)x, \end{aligned}$$

where the last step, $R(\lambda; T)(\lambda I - T)R(\mu; T)x = R(\mu; T)x$, is justified since $R(\mu; T)x \in \mathfrak{D}$. It follows that $R(\lambda; T)$ is a solution of the resolvent equation, belonging to $\mathfrak{E}(\mathfrak{X})$ in its domain of existence $\rho(T)$. It is no longer a resolvent in the sense of Definition 5.8.1, but it is a pseudo-resolvent and all results in the preceding discussion which apply to arbitrary solutions of the resolvent equation also apply to $R(\lambda; T)$.

Since $R(\lambda; T)$ is locally holomorphic, $\rho(T)$ is still an open set and $\sigma(T)$ is closed. Further, $\sigma(T)$ cannot be vacuous, since this would make $R(\lambda; T)$ holomorphic in the extended plane so that $R(\lambda; T)$ would be a constant and hence by Theorem 5.9.3, $R(\lambda; T) = Z$ where $Z^2 = \Theta$. But $(\lambda I - T)Zx = x$ for all x is clearly impossible so that $\sigma(T)$ is non-vacuous. It should be noted, however, that in the unbounded case $R(\lambda; T)$ may very well be an entire function of λ and $\sigma(T)$ reduce to the point at infinity. The following simple example illustrates this possibility:

(v) $\mathfrak{X} = C_0[0, 1]$, T takes $x(t)$ into $x'(t)$, $\mathfrak{D}$ is the subspace of $C_0[0, 1]$ where $x'(t)$ exists and belongs to $C_0[0, 1]$. Here $R(\lambda; T)$ exists and takes $x(t)$ into $-\int_0^t e^{\lambda(t-u)}x(u)\, du$ which is an entire function of λ.

The series expansions based on the resolvent equation hold also in the unbounded case and so does the Laurent resolution, Theorem 5.10.1. The con-

siderations of section 5.11 carry over to a considerable extent as will be seen by the two following theorems.

THEOREM 5.14.2. *Let $\sigma_1, \sigma_2, \cdots, \sigma_k$ be bounded spectral sets of the operator T and $\sigma_\alpha \cap \sigma_\beta = \emptyset$, $\alpha \neq \beta$. Define*

$$J_\alpha = \frac{1}{2\pi i} \int_{\Gamma_\alpha} R(\zeta; T)\, d\zeta,$$

where Γ_α is an oriented envelope of σ_α. Then

$$J_\alpha^2 = J_\alpha, \qquad J_\alpha J_\beta = \Theta, \qquad \alpha \neq \beta, \qquad J_\alpha \neq \Theta, I.$$

Setting

$$T_\alpha = TJ_\alpha, \qquad R_\alpha(\lambda; T) = J_\alpha R(\lambda; T),$$

then $T_\alpha \in \mathfrak{E}(\mathfrak{X})$ and for $\alpha \neq \beta$

$$T_\alpha T_\beta = \Theta, \qquad T_\alpha R_\beta(\lambda; T) = R_\beta(\lambda; T) T_\alpha = \Theta, \qquad R_\alpha(\lambda; T) R_\beta(\lambda; T) = \Theta,$$

$$(J_\alpha \lambda - T_\alpha) R_\alpha(\lambda; T) = R_\alpha(\lambda; T)(J_\alpha \lambda - T_\alpha) = J_\alpha.$$

Furthermore, the spectrum of T_α is σ_α in addition to $\lambda = 0$ and $R_\alpha(\lambda; T)$ is holomorphic in every domain of the λ-plane which does not contain any points of σ_α. Finally

$$R_\alpha(\lambda; T) = J_\alpha \lambda^{-1} + \sum_1^\infty T_\alpha^n \lambda^{-n-1}$$

for $|\lambda| > \gamma_\alpha = \lim_{n\to\infty} \| T_\alpha^n \|^{1/n}$.

PROOF. We have merely to examine the argument of Theorem 5.11.1 to see what carries over to the new situation. We note first that the proofs of $J_\alpha^2 = J_\alpha$ and $J_\alpha J_\beta = \Theta$ carry over unchanged. The relations $(\lambda I - T)R(\lambda; T)x = x$ for all x and $R(\lambda; T)(\lambda I - T)x = x$ in $\mathfrak{D}$ show that

$$TJ_\alpha x = \frac{1}{2\pi i} \int_{\Gamma_\alpha} \zeta R(\zeta; T) x\, d\zeta \qquad \text{for all } x$$

and $J_\alpha Tx$ is given by the same expression when $x \in \mathfrak{D}$. Hence J_α and T commute when operating on elements of $\mathfrak{D}$ and $T_\alpha = TJ_\alpha$ is a bounded operator so that $T_\alpha \in \mathfrak{E}(\mathfrak{X})$. Further J_α and $R(\lambda; T)$ commute because $R(\lambda; T)$ and $R(\zeta; T)$ commute. The orthogonality properties of T_α and $R_\alpha(\lambda; T)$ then follow from those of J_α. The quasi-resolvent relations follow in one direction by operating on both sides of $(\lambda I - T)R(\lambda; T)x = x$ by $J_\alpha^2 = J_\alpha$ and in the other by applying the operator $J_\alpha R(\lambda; T)(\lambda I - T)$ to $J_\alpha x$. The rest of the proof goes as above.

THEOREM 5.14.3. *Let $R(\lambda; T)$ be holomorphic save for the isolated singularities $\lambda_1, \lambda_2, \cdots, \lambda_k, \infty$. Then there exist k idempotents $J_1, J_2, \cdots, J_k$ where*

$$J_\alpha = \frac{1}{2\pi i}\int_{\Gamma_\alpha} R(\zeta; T)\, d\zeta,$$

Γ_α being a small circle about $\zeta = \lambda_\alpha$, such that

$$R(\lambda; T) = \sum_1^k R_\alpha^-(\lambda; T) + R^+(\lambda; T),$$

$$R_\alpha^-(\lambda; T) = J_\alpha R(\lambda; T) = J_\alpha(\lambda - \lambda_\alpha)^{-1} + \sum_1^\infty (T_\alpha^-)^n(\lambda - \lambda_\alpha)^{-n-1},$$

$$R^+(\lambda; T) = \sum_0^\infty (T_\infty)^{n+1}(-\lambda)^n.$$

Further

$$R_\alpha^-(\lambda; T)R^+(\lambda; T) = R^+(\lambda; T)R_\alpha^-(\lambda; T) = \Theta, \qquad T_\alpha^- T_\infty = T_\infty T_\alpha^- = \Theta,$$

$$R_\alpha^-(\lambda; T)R_\beta^-(\lambda; T) = \Theta, \qquad T_\alpha T_\beta = \Theta, \text{ for } \alpha \neq \beta$$

Here $T_\alpha^- = (T - \lambda_\alpha I)J_\alpha$ and T_∞ are quasi-nilpotents.

PROOF. The preceding theorem shows that the J_α's are idempotents which are mutually orthogonal. Further if Γ is the circle $|\zeta| = \gamma > \max|\lambda_i|$ and

$$J = \frac{1}{2\pi i}\int_\Gamma R(\zeta; T)\, d\zeta$$

then $J = \sum_1^k J_\alpha$ and $J^2 = J$. The $R_\alpha^-(\lambda; T)$ exist, are orthogonal to each other, and are quasi-resolvents of $T_\alpha = TJ_\alpha$. The only singular point of $R_\alpha^-(\lambda; T)$ is λ_α and since R_α^- vanishes at infinity, its expansion in powers of $(\lambda - \lambda_\alpha)$ must have the form indicated. Furthermore

$$T_\alpha^- = \frac{1}{2\pi i}\int_{\Gamma_\alpha} (\zeta - \lambda_\alpha)R(\zeta; T)\, d\zeta = TJ_\alpha - \lambda_\alpha J_\alpha.$$

To complete the argument, one uses the Laurent resolution of $R(\lambda; T)$ for the annulus $\gamma < |\lambda| < \infty$, obtaining

$$R(\lambda; T) = R^-(\lambda; T) + R^+(\lambda; T)$$

where

$$R^-(\lambda; T) = JR(\lambda; T) = \sum_1^k J_\alpha R(\lambda; T) = \sum_1^k R_\alpha^-(\lambda; T),$$

$$R^+(\lambda; T) = (I - J)\, R(\lambda; T) = \sum_0^\infty (T_\infty)^{n+1}(-\lambda)^n$$

with

$$T_\infty = \frac{1}{2\pi i}\int_\Gamma R(\zeta; T)\,\frac{d\zeta}{\zeta}.$$

The orthogonality of $R^+(\lambda; T)$ and $R^-_\alpha(\lambda; T)$ follows from that of $I - J$ and J_α. This completes the proof.

Finally we observe that Theorem 5.14.1 holds also for unbounded operators T; the proof carries over unchanged and the remark $Jx = x$ for characteristic elements is also valid.

References. Bochner and Phillips [1], Dunford [7, 8], Lorch [2, 3], Nagumo [1], Rickart [1], Šilov [1], Taylor [2, 5, 7].

4. Functions on the Algebra to Itself

5.15. Preliminaries. There are some new aspects of the theory of functions on vectors to vectors which come to the foreground when the underlying spaces are (B)-algebras. These new aspects get most of the attention in the following; after some preliminary remarks on contour integration, analyticity in commutative (B)-algebras, and the second resolvent equation, we pass over to the main question, viz. the extension of holomorphic scalar functions to analytic vector functions.

Let $\mathfrak{D}$ be an open connected subset of the complex non-commutative (B)-algebra $\mathfrak{B}$ with unit element e. Let $y = f(x)$ be a function on $\mathfrak{B}$ to itself with domain $\mathfrak{D}$. If $f(x)$ is continuous in $\mathfrak{D}$, Riemann integrals may be defined in an obvious manner. Let Γ be a rectifiable arc in $\mathfrak{D}$. By this we mean that Γ is given by an equation $x = x(\xi)$, $0 \leqq \xi \leqq 1$, where $x(\xi)$ is continuous and of strongly bounded variation in the sense of Definition 3.4.4. We then define

$$\int_\Gamma f(x)\cdot dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x(\eta_{n,k}))[x(\xi_{n,k}) - x(\xi_{n,k-1})],$$

where $\max_k (\xi_{n,k} - \xi_{n,k-1}) \to 0$. The existence of the integral is established in the usual manner and it has the properties of linearity and boundedness which are to be expected. In particular,

$$\left\| \int_\Gamma f(x)\cdot dx \right\| \leqq \max \|f(x)\|\, l(\Gamma), \tag{5.15.1}$$

where $l(\Gamma)$ is the length of Γ, that is, the strong total variation of $x(\xi)$ in [0, 1]. Since $\mathfrak{B}$ is non-commutative, there is also an integral $\int_\Gamma dx\cdot f(x)$ which is ordinarily distinct from $\int_\Gamma f(x)\cdot dx$.

This concept of the integral was introduced by E. R. Lorch [3] for a commutative (B)-algebra. For this case he also introduced the following definition of differentiabiliy and analyticity:

Definition 5.15.1. *Let $\mathfrak{B}$ be a commutative complex (B)-algebra with a unit element. A function $f(z)$ whose domain $\mathfrak{D}$ and range $\mathfrak{R}$ are in $\mathfrak{B}$ is said to have*

a derivative $f'(z_0)$ *at* $z = z_0$ *if for each* $\epsilon > 0$ *a* $\delta > 0$ *can be found such that for all* h *in* $\mathfrak{B}$ *with* $\|h\| < \delta$

$$\|f(z_0 + h) - f(z_0) - hf'(z_0)\| < \epsilon \|h\|. \tag{5.15.2}$$

If $f(z)$ *has a derivative everywhere in* $\mathfrak{D}$, *then it is analytic in* $\mathfrak{D}$.

A function analytic according to this definition is continuous and F-differentiable and hence analytic in the sense of Definition 4.5.2. By Theorem 4.5.1 its first variation $\delta f(z; h) = hf'(z)$ is an analytic function of z. We shall show that $f'(z)$ is also analytic in the sense of Lorch. Since $f(z)$ has second variations,

$$\lim_{\zeta \to 0} \frac{1}{\zeta}[f'(z + \zeta k) - f'(z)] \equiv f_2(z, k)$$

exists and

$$\delta^2 f(z; h, k) = hf_2(z, k) = kf_2(z, h)$$

since the second variation is symmetric in h and k. Putting $h = e$, we see that $f_2(z, k) = kf_2(z, e) \equiv kf''(z)$ and

$$\delta^2 f(z; h, k) = hkf''(z).$$

Since $\delta^2 f(z; h, k)$ is an F-differential we have

$$\|\delta f(z + h; k) - \delta f(z; k) - \delta^2 f(z; h, k)\| = o(\|h\|)$$

which for $k = e$ reduces to

$$\|f'(z + h) - f'(z) - hf''(z)\| = o(\|h\|).$$

Thus $f''(z)$ is actually the derivative of $f'(z)$ and the latter is analytic in the sense of Lorch. From this one concludes that $f(z)$ has derivatives of all orders with $\delta^n f(z; h) = f^{(n)}(z)h^n$, and may be expanded in a convergent Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - a)^n, \qquad n!\, a_n = f^{(n)}(a), \tag{5.15.3}$$

about any point a in $\mathfrak{D}$. Conversely, such a power series defines a Lorch analytic function of z in the interior of its set of convergence. In the notation of Definition 4.7.1 we have $\rho_a = \rho_u = 1/\mu$ where

$$\mu = \limsup \|a_n\|^{1/n}.$$

We note that every term of (5.15.3) is an F-power in the variable $z - a$ and a Lorch analytic function of z. Conversely, a homogeneous polynomial of degree n in z is necessarily of the form az^n if it is analytic in the sense of Lorch.

From the last observation we may conclude that *not every F-differentiable function on a commutative (B)-algebra to itself is Lorch analytic.* This follows, for instance, from the fact that such an algebra may contain other continuous linear functions of z than the multiples of z itself.

Examples can be found from the theory of functions of two complex variables. The space Z_2 of elements $\zeta = (\zeta_1, \zeta_2)$ can be made into a commutative (B)-algebra by defining

$$(\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2),\ \alpha(\beta_1, \beta_2) = (\alpha\beta_1, \alpha\beta_2),$$

$$(\alpha_1, \alpha_2)\cdot(\beta_1, \beta_2) = (\alpha_1\beta_1, \alpha_2\beta_2),\ \|(\alpha_1, \alpha_2)\| = \max[|\alpha_1|, |\alpha_2|].$$

Here $f(\zeta) = (\zeta_2, \zeta_1)$ is an F-differentiable function of ζ which is not Lorch analytic.

5.16. The second resolvent equation. As a preparation for the problem of extending holomorphic scalar functions to a Banach algebra, we study the resolvent $R(\lambda; x) = (\lambda e - x)^{-1}$ as a function of x.

THEOREM 5.16.1. *Let x and y belong to a complex Banach algebra $\mathfrak{B}$, having a unit element e, and suppose that $\lambda \in \rho(x) \cap \rho(y)$ so that $R(\lambda; x)$ and $R(\lambda; y)$ exist. Then*

$$R(\lambda;x) - R(\lambda;y) = R(\lambda;x)(x - y)R(\lambda;y). \tag{5.16.1}$$

PROOF. Formula (5.16.1), hereinafter referred to as the *second resolvent equation*, is obtained from the identity

$$(\lambda e - x)[R(\lambda; x) - R(\lambda; y)](\lambda e - y) = x - y$$

upon multiplication by $R(\lambda; x)$ on the left and $R(\lambda; y)$ on the right.

Let us consider the functional equation

$$R(x) - R(y) = R(x)(x - y)R(y) \tag{5.16.2}$$

on $\mathfrak{B}$ to itself. The following existence theorem should be compared with Theorem 5.9.2.

THEOREM 5.16.2. *Let $R(x)$ be a single-valued function on a Banach algebra $\mathfrak{B}$ to itself, satisfying (5.16.2) for all values of x and y in a domain $\mathfrak{D}$ of $\mathfrak{B}$. Then $R(x)$ is analytic in $\mathfrak{D}$. A necessary and sufficient condition for the existence of an element c of $\mathfrak{B}$ such that $(c - x)R(x) = R(x)(c - x) = e$ for all x in $\mathfrak{D}$ is that $R(x)$ be a regular element of $\mathfrak{B}$ for at least one value of x in $\mathfrak{D}$.*

PROOF. If $a \in \mathfrak{D}$ then

$$R(x)[e - (x - a)R(a)] = R(a). \tag{5.16.3}$$

By Theorem 5.2.1 the second factor on the left has an inverse if $\|(x - a)R(a)\| < 1$. Multiplication on the right of both sides by this inverse yields

$$R(x) = R(a)\left\{e + \sum_{n=1}^{\infty}[(x - a)R(a)]^n\right\}. \tag{5.16.4}$$

If $R(a) = \theta$, then obviously $R(x) \equiv \theta$. Excluding this trivial case, one sees that the power series is absolutely convergent in the sphere $\|x - a\| < \|R(a)\|^{-1}$ and is uniformly convergent in any concentric sphere of smaller radius. In

this sphere $R(x)$ is consequently analytic; a being arbitrary it follows that $R(x)$ is analytic in $\mathfrak{D}$.

If there exists a fixed element c of $\mathfrak{B}$ such that $(c - x)R(x) = R(x)(c - x) \equiv e$ in $\mathfrak{D}$, then $R(x)$ obviously has an inverse everywhere in $\mathfrak{D}$ so that the stated condition is necessary. Suppose conversely that $[R(a)]^{-1}$ exists, $a \in \mathfrak{D}$. From (5.16.3) one infers that $R(x)\{[R(a)]^{-1} + a - x\} \equiv e$ in $\mathfrak{D}$ and, interchanging x and a in (5.16.3), one sees that $[R(a)]^{-1} + a - x$ is also the left inverse of $R(x)$. This completes the proof of the theorem.

On the other hand, if $c - x$ is a regular element of $\mathfrak{B}$ for every x in some domain $\mathfrak{D} \subset \mathfrak{B}$, then $(c - x)^{-1}$ is obviously a solution of (5.16.2) in $\mathfrak{D}$. If $c = \lambda e$, this solution will also satisfy the first resolvent equation (5.9.2) with respect to λ.

The assumption in Theorem 5.16.2 that $[R(x)]^{-1}$ shall exist for at least one x in $\mathfrak{D}$ is not redundant. Indeed formula (5.16.4) defines a solution of equation (5.16.2) in the sphere $\| x - a \| < \| R(a) \|^{-1}$, no matter how $R(a)$ is chosen, and this solution has an inverse in its domain of definition if and only if $R(a)$ has an inverse.

We emphasize that for every fixed b the series

$$(5.16.5) \qquad R(x) = b \sum_{n=0}^{\infty} [(x - a)b]^n$$

defines a solution of the second resolvent equation, analytic in the sphere $\| x - a \| < \| b \|^{-1}$, and that every solution of the equation which is analytic in a neighborhood of $x = a$ is of this form.

5.17. Extension of holomorphic scalar functions. Among the analytic functions on a complex Banach algebra to itself an important class is formed by *the functions which become analytic in the classical sense on the subalgebra of complex numbers*. This class has been studied from different angles by N. Dunford, I. Gelfand, E. R. Lorch, and A. E. Taylor. We shall attempt to fit the theory of these functions into the general theory of analytic functions on $\mathfrak{B}$ to $\mathfrak{B}$.

The question may be looked upon as an *extension problem*. The algebra $\mathfrak{C}$ of complex numbers is extended to a complex Banach algebra $\mathfrak{B}$ so that $\mathfrak{C}$ becomes embedded in $\mathfrak{B}$. Is it possible simultaneously to extend holomorphic functions $f(\lambda)$ on $\mathfrak{C}$ to analytic functions $f(x)$ on $\mathfrak{B}$ in such a manner that $f(\lambda e) = f(\lambda)e$? In this formulation the question does not have a unique solution inasmuch as an analytic function $f(x)$ on $\mathfrak{B}$ to $\mathfrak{B}$ may vanish for all x in $\mathfrak{C}e$ without vanishing identically.

As an illustration, let $\mathfrak{B}$ be a complex matrix algebra, A a matrix such that $A^k = \Theta$ but $A^{k-1} \neq \Theta$, then any non-trivial product of k factors A and n factors X is a homogeneous polynomial in X of degree n which vanishes when $X = \lambda E$ without vanishing identically. There are consequently infinitely many analytic functions which vanish on the complex plane without vanishing identically and the extension problem becomes indeterminate.

It is possible, however, to define a unique *principal extension* in a perfectly natural manner and the resulting class of analytic functions has simple properties. Actually there are several different procedures which lead to the same result. We shall start with the local point of view which is less *ad hoc* than the methods in the large.

DEFINITION 5.17.1. *If $f(\lambda)$ is holomorphic in the circle $|\lambda - \lambda_0| < \rho$, then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\lambda_0)}{n!} (x - \lambda_0 e)^n \tag{5.17.1}$$

is by definition the principal extension of $f(\lambda)$ in the sphere $\| x - \lambda_0 e \| < \rho$.

THEOREM 5.17.1. *If $f(\lambda)$ is holomorphic in a domain Δ of the complex plane and if $\mathfrak{D}_\Delta$ is the union of all spheres $\| x - \lambda_0 e \| < \rho$ such that the circle $|\lambda - \lambda_0| < \rho$ is in Δ, then the principal extension of $f(\lambda)$ is uniquely determined by Definition 5.17.1 in the domain $\mathfrak{D}_\Delta$ and is an analytic function of x in $\mathfrak{D}_\Delta$.*

PROOF. We denote the power series in formula (5.17.1) by $f(x; \lambda_0 e)$. It is clearly an analytic function of x in the sphere of definition. Thus we have merely to verify that the various power series $f(x; \lambda_0 e)$ are elements of the same analytic function $f(x)$. Suppose that two circles $|\lambda - \lambda_1| < \rho_1$ and $|\lambda - \lambda_2| < \rho_2$ in Δ overlap and let λ_0 be a point common to both. The classical argument based on rearrangement of power series now shows that $f(x; \lambda_1 e) = f(x; \lambda_0 e)$ in the sphere $\| x - \lambda_0 e \| < \rho_1 - |\lambda_1 - \lambda_0|$ and in a concentric sphere $f(x; \lambda_2 e) = f(x; \lambda_0 e)$. It follows that the various elements $f(x; \alpha e)$, $\alpha \in \Delta$, are analytic continuations of each other and define uniquely an analytic function $f(x)$ in $\mathfrak{D}_\Delta$.

THEOREM 5.17.2. *Let $x \in \mathfrak{D}_\Delta$ and choose a $\lambda_0 \in \Delta$ such that the circle Γ_x : $|\lambda - \lambda_0| = \| x - \lambda_0 e \| + \epsilon$, $\epsilon > 0$, lies in Δ. Then*

$$f(x) = \frac{1}{2\pi i} \int_{\Gamma_x} f(\lambda) R(\lambda; x)\, d\lambda. \tag{5.17.2}$$

PROOF. The definition of $\mathfrak{D}_\Delta$ shows that it is always possible to find such a circle Γ_x. We apply formula (5.16.4) to $R(x) = R(\lambda; x)$, choosing $a = \lambda_0 e$. Since $R(\lambda; \lambda_0 e) = (\lambda - \lambda_0)^{-1} e$, the formula simplifies to

$$R(\lambda; x) = \sum_{n=0}^{\infty} (x - \lambda_0 e)^n (\lambda - \lambda_0)^{-n-1}. \tag{5.17.3}$$

Substituting this expression into (5.17.2), we see that the integral equals $f(x; \lambda_0 e) = f(x)$ and the theorem is proved.

Formula (5.17.3) shows that $\sigma(x)$, the spectrum of x, lies in the circle $|\lambda - \lambda_0| \leqq \| x - \lambda_0 e \|$, that is, $\sigma(x)$ is interior to Δ when $x \in \mathfrak{D}_\Delta$. But for the existence of the integral in (5.17.2) it is not essential that we integrate along the circle Γ_x; any closed contour in Δ surrounding $\sigma(x)$ will do. This indicates

that the integral has a meaning as long as $\sigma(x)$ lies in Δ and this may very well happen for x outside of $\mathfrak{D}_\Delta$. Before we can proceed to a study of the resolvent integral, a few observations of a mixed function theoretical and topological nature are required.

Let Φ be a closed bounded set in the complex λ-plane and $f(\lambda)$ a function holomorphic in a domain Δ containing Φ. An argument of the Heine-Borel type shows that we can find an open set Ω with the following properties (i) $\Phi \subset \Omega \subset \Delta$, (ii) Ω has a finite number of components Ω_μ, (iii) each Ω_μ is bounded by a finite number of simple closed rectifiable curves $\Gamma_{\mu\nu}$, and (iv) Ω has a positive distance from the boundary of Δ. We assign a positive orientation to each $\Gamma_{\mu\nu}$ in the usual manner and let $\Gamma = \bigcup \Gamma_{\mu\nu}$ be the boundary of Ω, the orientation of Γ being induced by that of $\Gamma_{\mu\nu}$. We call Γ *an oriented envelope of* Φ *with respect to* $f(\lambda)$. For $\lambda \in \Omega$ we have

$$f(\lambda) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - \lambda}\,d\zeta.$$

If $\lambda \in \Phi$, the integral is independent of the choice of Γ. The proof can be reduced to the classical elementary case by a somewhat laborious discussion of the different topological possibilities.

We shall also need the following theorem concerning spectra.

THEOREM 5.17.3. *The spectrum of x is a continuous function of x.*

REMARK. *We say that $\sigma(x)$ is continuous at $x = a$ if, given any open set Ω containing $\sigma(a)$, there is a positive ϵ such that $\sigma(x) \subset \Omega$ for $\| x - a \| < \epsilon$.*

PROOF. Let a be given and let Ω be an open set containing $\sigma(a)$. The usual Heine-Borel argument shows that we may cover the closed bounded set $\sigma(a)$ by a finite number of circles with radius 2δ where δ is so chosen that the union of the interiors of the circles, Ω_δ say, has the properties $\sigma(a) \subset \Omega_\delta \subset \Omega$, $d[\sigma(a), \Omega_\delta^*] > \delta$, and Ω_δ^* is the complement of Ω_δ. The interior of Ω_δ^* consists of a finite number of components, the closure of each of which lies in $\rho(a)$. It follows that $R(\lambda; a)$ is holomorphic everywhere in Ω_δ^* and there exists a finite $M = M(a, \delta)$ such that $\| R(\lambda; a) \| \leqq M$ for $\lambda \in \Omega_\delta^*$. By formula (5.16.4), $R(\lambda; x)$ exists for every such λ provided $\| x - a \| < 1/M = \epsilon$. It follows that for any such x we have $\rho(x) \supset \Omega_\delta^*$ and $\sigma(x) \subset \Omega_\delta \subset \Omega$. This completes the proof.

We come now to the main extension theorem.

THEOREM 5.17.4. *Let $f(\lambda)$ be holomorphic in the domain Δ. Let $\mathfrak{S}(\Delta)$ be the open set of points x of $\mathfrak{B}$ such that $\sigma(x) \subset \Delta$ and let $\mathfrak{D}(\Delta)$ be the component of $\mathfrak{S}(\Delta)$ which contains the set Δe. For x in $\mathfrak{D}(\Delta)$ define*

$$(5.17.4) \qquad f(x) = \frac{1}{2\pi i}\int_{\Gamma_x} f(\lambda)\,R(\lambda; x)\,d\lambda,$$

where Γ_x is any oriented envelope of $\sigma(x)$ with respect to $f(\lambda)$. Then $f(x)$ is analytic in $\mathfrak{D}(\Delta)$ and coincides with the principal extension of $f(\lambda)$ in $\mathfrak{T}_\Delta$.

PROOF. That $\mathfrak{G}(\Delta)$ is an open set follows from the preceding theorem. The component $\mathfrak{D}(\Delta)$ and the integral are then well defined. That $\mathfrak{D}(\Delta)$ contains the domain $\mathfrak{D}_\Delta$ of Theorem 5.17.1 follows from the remarks after the proof of Theorem 5.17.2 and the latter shows that the integral represents the principal extension of $f(\lambda)$ in $\mathfrak{D}_\Delta$. It remains to show that $f(x)$ is analytic in the larger domain $\mathfrak{D}(\Delta)$. For this purpose, let $a \in \mathfrak{D}(\Delta)$ and choose an open sphere $\mathfrak{S}(a): \| x - a \| < \rho$ so small that Φ, the closure of $\bigcup_x \sigma(x)$, $x \in \mathfrak{S}(a)$, is in Δ. For these values of x we may replace Γ_x by Γ, a fixed oriented envelope of Φ with respect to $f(\lambda)$. Since $R(\lambda; a)$ is holomorphic on each component $\Gamma_{\mu\nu}$ of Γ, there exists a finite quantity $M(a)$ with $\| R(\lambda; a) \| \leqq M(a)$ everywhere on Γ. Formula (5.16.4) shows that for $\lambda \in \Gamma$

$$R(\lambda; x) = R(\lambda; a) \sum_{n=0}^{\infty} [(x - a)R(\lambda; a)]^n,$$

the series being absolutely convergent for $\| x - a \| \leqq \rho_0 < \min \{\rho, 1/M(a)\}$, the convergence being uniform with respect to x in the sphere as well as with respect to λ on Γ. Termwise integration gives

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_\Gamma f(\lambda)R(\lambda; a)[(x - a)R(\lambda; a)]^n \, d\lambda. \tag{5.17.5}$$

This is an abstract power series in $(x - a)$ which is absolutely and uniformly convergent when $\| x - a \| \leqq \rho_0$. It follows that $f(x)$ is analytic in this sphere and consequently everywhere in $\mathfrak{D}(\Delta)$. This completes the proof and justifies

DEFINITION 5.17.2. *The principal extension of $f(\lambda)$ is defined by formula* (5.17.4) *in* $\mathfrak{D}(\Delta)$.

The use of formula (5.17.4) in matrix theory and operational calculus is of old standing and goes back to H. Poincaré (1900) in a memoir devoted to the theory of continuous groups. F. Riesz (1913) introduced similar devices in Hilbert space theory. In one form or another it is basic in L. Fantappiè's theory of analytic functionals as well as in recent investigations of commutative normed vector rings by I. Gelfand and E. R. Lorch, of spectral theory by N. Dunford, and of analysis in complex Banach spaces by A. E. Taylor. For further historical remarks we refer to the expository papers of Dunford [7] and of Taylor [7].

Formula (5.17.4) presupposes that the Banach algebra has a unit element. It is possible, however, to give an alternate definition of $f(x)$ (when $f(0) = 0$) which is equivalent to (5.17.4) when the latter applies, but which holds in any complex Banach algebra. See further Chapter XXII, especially section 22.9.

The interpretation of the symbol $f(x)$ when $f(\lambda)$ is holomorphic in the sense of Cauchy and x is an element of a (B)-algebra (usually an algebra of endomorphisms $\mathfrak{E}(\mathfrak{X})$) varies in recent investigations. Lorch has emphasized with justice that analyticity does not reside in the symbol $f(\cdot)$ and that the term "analytic function of x" should be reserved for differentiable functions of x. The functions defined by formula (5.17.4) satisfy this

requirement, if one is satisfied with the existence of Fréchet differentials and analyticity in the sense of Definition 4.5.2. The more restrictive definition of analyticity due to Lorch works well in the commutative case for which it is constructed, but loses its significance in a non-commutative algebra.

Formula (5.17.4) may be regarded as a natural generalization of the well known formula of Cauchy. The formulas for the derivatives also admit of generalizations which may be read off from (5.17.5) together with Theorem 4.3.10:

$$(5.17.6) \qquad \delta^n f(x; h) = \frac{n!}{2\pi i}\int_\Gamma f(\lambda)R(\lambda; x)[hR(\lambda; x)]^n\, d\lambda.$$

The correspondence between the functions $f(\lambda)$, holomorphic in a domain Δ, and their principal extensions $f(x)$ in $\mathfrak{D}(\Delta)$ has been investigated by I. Gelfand [4] and A. E. Taylor [7]. We shall prove

THEOREM 5.17.5. *Let Δ be a bounded domain of the complex plane and bounded by a finite number of Jordan curves or Jordan arcs or both, no two of which have more than a finite number of points in common. Let $\mathfrak{D}(\Delta)$ be the corresponding domain of $\mathfrak{B}$ as defined above. Let $\mathfrak{C}[\Delta]$ be the complex (B)-algebra of all functions $f(\lambda)$ holomorphic in Δ and continuous in the closure $\bar{\Delta}$ of Δ, with the ordinary definition of the arithmetic operations and with $\| f \|_\Delta = \sup | f(\lambda) |$, $\lambda \in \Delta$. Further let $\mathfrak{B}[\Delta]$ be the complex algebra of functions $F(x)$ analytic in $\mathfrak{D}(\Delta)$ and having values in $\mathfrak{B}$, the arithmetic operations being defined as in $\mathfrak{B}$.*

There exists an isomorphic mapping: $f(\lambda) \to f(x)$ of $\mathfrak{C}[\Delta]$ on a subalgebra $\mathfrak{B}_0[\Delta]$ of $\mathfrak{B}[\Delta]$ such that (i) $\lambda \to x$ *and* (ii) $\| f_n - f \|_\Delta \to 0$ *implies that* $\| f_n(x) - f(x) \| \to 0$ *locally uniformly in $\mathfrak{D}(\Delta)$. This mapping is unique and is defined by* (5.17.4)

REMARK. For the meaning of the terms *homomorphism* and *isomorphism* used below, see Definition 22.10.2.

PROOF. We start by proving that the mapping defined by (5.17.4) has the required properties. That the mapping takes $f(\lambda) = \lambda$ into $f(x) = x$ follows from Definition 5.17.1. It is clear that the correspondence is linear so in order to prove that the mapping is a homomorphism it is sufficient to prove that products go into products. This may be proved with the aid of the first resolvent relation and formula (5.17.4), but it is really simpler to use a local argument plus analytic continuation. In fact, if $f(\lambda)$, $g(\lambda)$, $h(\lambda) \in \mathfrak{C}[\Delta]$ and $h(\lambda) \equiv f(\lambda)g(\lambda)$, then in any circle $|\lambda - \lambda_0| < \rho$ in Δ we have an identity between power series

$$\sum_{n=0}^{\infty} f_{n0}(\lambda - \lambda_0)^n \cdot \sum_{n=0}^{\infty} g_{n0}(\lambda - \lambda_0)^n \equiv \sum_{n=0}^{\infty} h_{n0}(\lambda - \lambda_0)^n$$

with obvious notation. By Definition 5.17.1 this gives

$$\sum_{n=0}^{\infty} f_{n0}(x - \lambda_0 e)^n \cdot \sum_{n=0}^{\infty} g_{n0}(x - \lambda_0 e)^n \equiv \sum_{n=0}^{\infty} h_{n0}(x - \lambda_0 e)^n$$

in the sphere $\| x - \lambda_0 e \| < \rho$, that is $h(x) \equiv f(x)g(x)$. By Theorem 4.6.1, this identity must hold everywhere in $\mathfrak{D}(\Delta)$. Hence the mapping is a homomorphism.

In order to prove that the homomorphism is actually an isomorphism, we have to show that the correspondence is one-to-one or that $f(x) \equiv \theta$ implies $f(\lambda) \equiv 0$. Suppose that for a particular choice of $f(\lambda)$ in $\mathfrak{C}[\Delta]$ we have $f(x) = \theta$ for all x in $\mathfrak{D}(\Delta)$. But if $\lambda \in \Delta$ than $\lambda e \in \mathfrak{D}(\Delta)$ and $f(\lambda e) = f(\lambda)e$ so that in particular $f(\lambda) \equiv 0$. Thus the mapping is an isomorphism.

Suppose that $f(\lambda) \in \mathfrak{C}[\Delta]$ and is bounded away from zero in some domain $\Delta_0 \subset \Delta$. Let $\mathfrak{C}[\Delta_0]$ and $\mathfrak{D}(\Delta_0)$ bear the same relation to Δ_0 as $\mathfrak{C}[\Delta]$ and $\mathfrak{D}(\Delta)$ have to Δ. Then $f(\lambda)$ and $[f(\lambda)]^{-1}$ belong to $\mathfrak{C}[\Delta_0]$ so that the corresponding functions $f(x)$ and $[f(x)]^{-1}$ exist for $x \in \mathfrak{D}(\Delta_0)$. Since $f(x)$ has an inverse for such values of x, it follows, in particular, that $f(x) \neq \theta$. Necessary and sufficient conditions on x in order that $f(x) = \theta$ have been given by N. Dunford [7, p. 644, the *minimal equation theorem*].

To prove the continuity, we proceed as in the proof of Theorem 5.17.4. We choose a point $a \in \mathfrak{D}(\Delta)$, a small sphere $\mathfrak{S}(a)$ with center at a, and let Φ be the closure of $\bigcup_x \sigma(x)$, $x \in \mathfrak{S}(a)$. In the representation of the functions $\{f_n(x)\}$ corresponding to the given Cauchy sequence $\{f_n(\lambda)\}$, we may replace Γ_x by a fixed contour Γ which is an oriented envelope of Φ with respect to all functions $f_n(\lambda)$. As above we have $\| R(\lambda; a) \| \leqq M(a)$ for λ on Γ and $\| R(\lambda; x) \| \leqq K(a)$ for $\| x - a \| \leqq \rho_0 < \min \{\rho, [M(a)]^{-1}\}$, λ on Γ, where $K(a) \leqq M(a)[1 - \rho_0 M(a)]^{-1}$. It follows that

$$\| f_n(x) - f(x) \| \leqq \frac{1}{2\pi} \| f_n - f \|_\Delta K(a)L(\Gamma), \qquad \| x - a \| \leqq \rho_0,$$

where $L(\Gamma)$ is the length of Γ, so that $f_n(x)$ converges locally uniformly to $f(x)$. This shows that the mapping defined by (5.17.4) has all the desired properties.

It remains to show that the mapping is unique. In order to prove this we observe first that if $\mathfrak{J}$ is any isomorphic mapping with the stated properties, then $\mathfrak{J}$ must map the zero and the unit elements of $\mathfrak{C}[\Delta]$ upon the zero and the unit elements of $\mathfrak{B}_0[\Delta]$, that is, $0 \to \theta$ and $1 \to e$. Further, $\mathfrak{J}$ maps the polynomial $P(\lambda) = \sum_{k=0}^n \alpha_k \lambda^k$ upon $P(x) = \alpha_0 e + \sum_{k=1}^n \alpha_k x^k$. Moreover, if α is not in $\bar{\Delta}$, then $(\alpha - \lambda)^{-1} \in \mathfrak{C}[\Delta]$ and $\alpha e - x$ is regular for x in $\mathfrak{D}(\Delta)$ so that $R(\alpha; x)$ exists. Since products go into products, $\mathfrak{J}$ must take $(\alpha - \lambda)^{-1}$ into $R(\alpha; x)$. This, however, agrees with the image under the principal extension; consequently $\mathfrak{J}$ will map each rational function whose poles are outside of $\bar{\Delta}$ on its principal extension as defined by (5.17.4). We can now appeal to an extension of the Runge theorem due to J. L. Walsh [**1**, p. 47]. Under the present assumptions on Δ, each $f(\lambda) \in \mathfrak{C}[\Delta]$ is the limit of a sequence of rational functions with poles outside of $\bar{\Delta}$, the convergence being uniform in $\bar{\Delta}$, and if Δ is simply-connected the rational functions may be taken as polynomials. In other words, the rational functions of $\mathfrak{C}[\Delta]$ are dense in this space. From the fact that $\mathfrak{J}$ agrees with the mapping defined by (5.17.4) in a dense set of $\mathfrak{C}[\Delta]$ together with the continuity assumptions, we conclude that $\mathfrak{J}$ is identical with the principal mapping everywhere in $\mathfrak{C}[\Delta]$ so the correspondence is unique.

There are several remarks which should be appended to the preceding definitions and theorems. The interpretation of $f(x)$ as the principal extension of $f(\lambda)$ appears to be new. A corresponding extension to (B)-algebras without unit element will be given in section 22.9.

If $\mathfrak{B}$ is a commutative (B)-algebra, $f(x)$ is not merely the principal extension of $f(\lambda)$ in $\mathfrak{D}(\Delta)$, but it is the only extension from scalars to $\mathfrak{B}$ which is analytic in the more stringent sense of Lorch, Definition 5.15.1. This is seen as follows. Suppose that $F(x)$ is Lorch analytic in a domain $\mathfrak{D}$ which intersects the complex plane in a domain Δ and that $F(\lambda e) = F(\lambda)e$ for $\lambda \in \Delta$ where $F(\lambda)$ is holomorphic in Δ. This requires that $F^{(n)}(\lambda e) = F^{(n)}(\lambda)e$ for every n and Definition 5.17.1 together with Theorem 4.6.1 show that $F(x)$ coincides with the principal extension of $F(\lambda)$. In this case formula (5.17.5) takes on the somewhat more familiar form

$$f(x) = \sum_{n=3}^{\infty} (x - a)^n \frac{1}{2\pi i} \int_\Gamma f(\lambda)[R(\lambda; a)]^{n+1} \, d\lambda. \tag{5.17.7}$$

This is a power series of the type occurring in formula (5.15.3). It consequently defines a Lorch analytic function, that is, *the principal extension of $f(\lambda)$ is analytic in the sense of Lorch if $\mathfrak{B}$ is a commutative complex (B)-algebra and it is the only extension of $f(\lambda)$ having this property.*

Gelfand also restricts himself to the commutative case. In his presentation $f(x)$ is defined by Theorem 5.17.5 (with slightly different assumptions) and he shows that if $\mathfrak{J}$ is a continuous isomorphic mapping, $f(\lambda) \to f(x)$, of holomorphic scalar functions onto abstract functions taking λ into x, then $f(x)$ is necessarily given by (5.17.4). He evidently regards $f(x)$ as an analytic function of x, but he does not state in what sense it is analytic nor does he study $f(x)$ as a function of x.

N. Dunford has extended this function concept in a manner which is of importance to operational calculus by allowing $f(\lambda)$ to be piecewise analytic. If $\sigma(x)$ is the union of a finite number of disjoint spectral sets σ_j in the sense of Definition 5.11.1, the set σ_j being contained in the component Ω_j of the open set Ω, then on the boundary $\bigcup_k \Gamma_{jk}$ of Ω_j, $f(\lambda) = f_j(\lambda)$, where $f_j(\lambda)$ is holomorphic in a domain containing Ω_j. Formula (5.17.4) still has a sense and is taken as the definition of $f(x)$. The analytic properties of such functions $f(x)$ have not been investigated; it is possible that $f(x)$ is piecewise analytic in some suitable sense.

Finally it should be observed that formula (5.17.4) defines an analytic function of x in each of the components of the open set $\mathfrak{G}(\Delta)$ of Theorem 5.17.4 and not merely in $\mathfrak{D}(\Delta)$. An example is given by the function x^{-1} which is defined and analytic in each of the components of the maximal group $\mathfrak{G}$ of section 5.3.

We end this discussion by proving some theorems which are all closely related to spectral theory. The first is the *spectral mapping theorem* of N. Dunford [**8**, p. 195].

THEOREM 5.17.6. *If $f(\lambda) \in \mathfrak{C}[\Delta]$ and $x \in \mathfrak{D}[\Delta]$, then $\sigma[f(x)] = f[\sigma(x)]$.*

PROOF. Let $\alpha \in \sigma(x)$ and define

$$g(\lambda) = [f(\alpha) - f(\lambda)](\alpha - \lambda)^{-1}, \qquad \lambda \in \Delta,$$

so that $g(\lambda) \in \mathfrak{C}[\Delta]$ and consequently

$$g(x)(\alpha e - x) = f(\alpha)e - f(x). \tag{5.17.8}$$

It follows that the right member cannot have an inverse and that $f(\alpha) \in \sigma[f(x)]$. Conversely, let $\mu \in \sigma[f(x)]$ and suppose that $\mu \notin f[\sigma(x)]$. We form $h(\lambda) = [\mu - f(\lambda)]^{-1}$ which is holomorphic in Δ save for poles, none of which belongs to $\sigma(x)$. We can then find a subdomain Δ_1 such that $\sigma(x) \subset \Delta_1 \subset \Delta$ and $h(\lambda)$ is holomorphic in Δ_1. Since $x \in \mathfrak{D}(\Delta_1)$, we have $h(x)[\mu e - f(x)] = e$ which contradicts the assumption that $\mu \in \sigma[f(x)]$.

If $\mathfrak{B}$ is a (B)-algebra of endomorphisms, $\mathfrak{E}(\mathfrak{X})$, even the finer structure of the spectrum is preserved under the mapping. We refer to Definition 15.5.1 for the terminology used below.

THEOREM 5.17.7. *If* $\mathfrak{B} = \mathfrak{E}(\mathfrak{X})$, *if* $f(\lambda) \in \mathfrak{C}[\Delta]$, *and if* $T \in \mathfrak{D}(\Delta)$, *then* $\mu I - f(T)$ *has the property* P_ν ($\nu = 1, 2,$ *or* 3) *if and only if the equation* $\mu = f(\lambda)$ *has a solution*, $\lambda = \alpha$, $\alpha \in \sigma(T)$, *such that* $\alpha I - T$ *has the same spectral property* P_ν.

PROOF. If $\alpha \in \sigma(T)$ and $g(\lambda)$ is defined as above then by formula (5.17.8) we have

$$[f(\alpha)I - f(T)]x = g(T)[\alpha I - T]x, \qquad x \in \mathfrak{X}.$$

This shows that the left member vanishes whenever $[\alpha I - T]x$ does, that is, property P_1 is preserved under the mapping. If, on the other hand, $\alpha I - T$ maps $\mathfrak{X}$ on a non-dense subspace $\mathfrak{X}_\alpha$, then it maps $g(T)[\mathfrak{X}]$ on a subspace of $\mathfrak{X}_\alpha$ so that $[f(\alpha)I - f(T)][\mathfrak{X}]$ is also non-dense. Finally, if the image of the unit sphere in $\mathfrak{X}$ under the operator $\alpha I - T$ is not bounded away from zero, then the same holds for the image under $f(\alpha)I - f(T)$. This proves that if $\alpha I - T$ has the spectral property P_ν then so does $f(\alpha)I - f(T)$. To prove the converse proposition, suppose that $\mu \in \sigma[f(T)]$. By the preceding theorem there is at least one $\alpha \in \sigma[T]$ with $f(\alpha) = \mu$. We may assume without restricting the generality that the equation $f(\lambda) = \mu$ has only a finite number of roots $\alpha_1, \alpha_2, \cdots, \alpha_n$ in Δ and that $f(\lambda) \neq \mu$ on the boundary of Δ. Set $P(\lambda) = \prod_1^n(\alpha_k - \lambda)$ and form

$$F(\lambda) = P(\lambda)[\mu - f(\lambda)]^{-1} \in \mathfrak{C}[\Delta].$$

Then $F(T)$ is well defined and

$$F(T)[\mu I - f(T)]x = P(T)x, \qquad x \in \mathfrak{X}.$$

It follows that any x which is annihilated by $\mu I - f(T)$ is also annihilated by $P(T)$. But this means that for at least one of the factors $\alpha_k I - T$ we can find a $y \neq \theta$ with $(\alpha_k I - T)y = \theta$. On the other hand, if $\mu I - f(T)$ maps $\mathfrak{X}$ on a non-dense linear subspace, then $P(T)[\mathfrak{X}]$ is also non-dense and this implies that $(\alpha_k I - T)[\mathfrak{X}]$ is non-dense for at least one k. Finally, if the image of the unit sphere under the operator $\mu I - f(T)$ is not bounded away from zero, then the same holds for the image under $P(T)$ and hence also for at least one of the factors $\alpha_k I - T$. It should be noted that $\mu I - f(T)$ has all the spectral properties of the operators $\alpha_k I - T$ with $f(\alpha_k) = \mu$ and no others. This completes the proof.

THEOREM 5.17.8. *If* $g(\lambda) \in \mathfrak{C}[\Delta]$ *and* $f(\mu)$ *is holomorphic for* μ *in* $g(\bar{\Delta})$ *so that* $f[g(\lambda)] \in \mathfrak{C}[\Delta]$, *then* $f[g(x)] \in \mathfrak{B}_0[\Delta]$ *and* $f[g(x)] = [f(g)](x)$.

PROOF. The existence of $[f(g)](x)$ for $x \in \mathfrak{D}(\Delta)$ is assured by Theorem 5.17.4 so we have merely to prove that $f[g(x)]$ exists and the equality. Since $\sigma[g(x)] = g[\sigma(x)]$ is a closed point set in $g(\Delta)$ and $f(\mu)$ is holomorphic in $g(\bar{\Delta})$, we can find a domain Δ_0 with $g[\sigma(x)] \subset \Delta_0 \subset g(\Delta)$ and $f(\mu) \in \mathfrak{C}[\Delta_0]$. Hence $f(y)$ exists and belongs to $\mathfrak{B}_0[\Delta_0]$ for every $y \in \mathfrak{D}[\Delta_0]$. In particular, we may take $y = g(x)$ so that $f[g(x)]$ exists for $x \in \mathfrak{D}(\Delta)$. To prove the equality we note that for $\mu \notin g(\bar{\Delta})$, the function $h(\lambda) \equiv [\mu - g(\lambda)]^{-1} \in \mathfrak{C}[\Delta]$ so that $h(x) = R(\mu; g(x))$ exists. Hence taking Γ and Γ_0 to be oriented envelopes of $g(\bar{\Delta})$ with respect to $f(\mu)$ and of $\sigma(x)$ with respect to $g(\lambda)$ respectively, we have

$$f[g(x)] = \frac{1}{2\pi i}\int_\Gamma f(\mu)R(\mu; g(x))\, d\mu = \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\Gamma_0} \frac{f(\mu)R(\lambda; x)}{\mu - g(\lambda)}\, d\lambda\, d\mu$$

$$= \frac{1}{2\pi i}\int_{\Gamma_0} f[g(\lambda)]R(\lambda; x)\, d\lambda = [f(g)](x).$$

The next theorem, which attaches to the ideas of Theorem 5.11.1, shows that a spectral resolution of x extends to the functions of x in $\mathfrak{B}_0[\Delta]$. We suppress the proof.

THEOREM 5.17.9. *Let* Δ *contain* $\lambda = 0$ *and let* $f(\lambda) \in \mathfrak{C}[\Delta]$. *If* $x \in \mathfrak{D}(\Delta)$ *and* $x_\alpha = j_\alpha x$, $\alpha = 1, 2, \cdots, k$, *then* $f(x_\alpha) \in \mathfrak{B}_0[\Delta]$,

$$f(x_\alpha) = (e - j_\alpha)f(0) + j_\alpha f(x), \qquad f(x) = \sum_1^k f(x_\alpha) - (k - 1)f(0)e.$$

5.18. The exponential function and the logarithm. The principal extension of $\exp(\lambda)$ is obviously given by

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad x \in \mathfrak{B}, \tag{5.18.1}$$

and this is the only extension which is analytic in the sense of Lorch if $\mathfrak{B}$ is commutative. If x and y commute

$$\exp(x)\exp(y) = \exp(x + y). \tag{5.18.2}$$

In particular one sees that $\exp(x)$ is always a regular element of $\mathfrak{B}$ having $\exp(-x)$ as its inverse. We have also the obvious

THEOREM 5.18.1. *The functional equation*

$$F(x + y) = F(x)F(y) \tag{5.18.3}$$

in a commutative (B)*-algebra is satisfied by*

$$F(x) = j\exp[P_1(x)], \tag{5.18.4}$$

where j *is an idempotent*, $j^2 = j$, *and* $P_1(x)$ *is an arbitrary homogeneous polynomial of degree one.*

We now proceed to define the logarithm of x which is obviously going to be infinitely many-valued. We start with a domain Δ in the complex plane which is bounded away from zero and infinity and in which $\arg \lambda$ is single-valued. Any determination of $\log \lambda$ will then be holomorphic in Δ and defines a corresponding determination of $\log x$ by

$$\log x = \frac{1}{2\pi i} \int_\Gamma \log \lambda \, R(\lambda; x) \, d\lambda \tag{5.18.5}$$

for $x \in \mathfrak{D}(\Delta)$. Here Γ is any simple closed rectifiable curve which contains Δ in its interior and $\lambda = 0$ in its exterior. If x is any regular element of $\mathfrak{B}$ whose spectrum does not separate $\lambda = 0$ from $\lambda = \infty$, then we may define $\log x$ by (5.18.5) because we can always find a domain Δ with the required properties. An alternate statement is that $\lambda = 0$ belongs to the principal component of the resolvent set of x.

The various determinations of $\log x$ defined by (5.18.5) obviously differ by multiples of $2\pi i e$. Since $\exp (2\pi i e) = e$ we have

$$\exp (x + 2\pi i e) = \exp x \tag{5.18.6}$$

for all x. Other periods will be discussed later. Using Theorem 5.17.8 we see that

$$\exp (\log x) = x, \qquad x \in \mathfrak{D}(\Delta), \tag{5.18.7}$$

for each of the above determinations of $\log x$. For the converse relation, let D be any bounded domain in the λ-plane such that no two points of D are congruent modulo $2\pi i$. The function $\mu = \exp \lambda$ maps D onto a domain Δ with the properties stated above in which $\log \lambda$ is holomorphic. Theorem 5.17.8 then shows that

$$\log [\exp x] = x + 2n\pi i e, \qquad x \in \mathfrak{D}(D), \tag{5.18.8}$$

where the integer n depends upon the determination of the logarithm. In particular, $n = 0$ if we choose the principal determination of $\log \lambda$ in (5.18.5).

In the commutative case Lorch defines the logarithm by

$$\log x = \int_e^x z^{-1} \, dz, \tag{5.18.9}$$

where the path joining e and x is an arbitrary rectifiable curve in $\mathfrak{G}_1$, the principal component of the maximal group. The resulting function $\log x$ is analytic in the sense of Lorch in $\mathfrak{G}_1$ which is the maximal domain of definition of the logarithm. See further section 22.3.

Lorch has also studied the periods of the exponential function (cf. section 22.5).

THEOREM 5.18.2. *If j is an idempotent, then* $\exp (2\pi i j) = e$ *and* $\exp (x + 2\pi i j) = \exp x$ *for each x commuting with j. If $\mathfrak{B}$ is commutative then $2\pi i j$ is a period of* $\exp x$ *and every period is of the form $2\pi i \sum_1^k n_\nu j_\nu$ where k is finite, the n's are*

integers, and the j's idempotents. In particular, if θ and e are the only idempotents of $\mathfrak{B}$, then $\exp x$ *is simply periodic.*

Proof. The reader can verify that $\exp(2\pi i j) = e$. Suppose that $\mathfrak{B}$ is commutative and that p is a period of $\exp x$. Then $\exp p = e$ and $\exp(\xi p)$ is a periodic function of ξ of period one. Hence (see section 11.2)

$$R(\lambda; p) = \int_0^\infty e^{-\lambda\xi} \exp(\xi p)\, d\xi = [1 - e^{-\lambda}]^{-1} \int_0^1 e^{-\lambda\xi} \exp(\xi p)\, d\xi,$$

where the three members are defined for $\lambda \in \rho(p)$, $\Re(\lambda) > 0$, and $\lambda \neq 2n\pi i$ respectively. This shows that $R(\lambda; p)$ is a meromorphic function of λ, with simple poles at a finite number of the integral multiples of $2\pi i$, say at $\lambda_\nu = 2\pi i n_\nu$, $\nu = 1, 2, 3, \cdots, k$. Theorem 5.11.2 applies and shows that the corresponding residues j_ν are idempotents, $e = \sum_1^k j_\nu$, $j_\mu j_\nu = \delta_{\mu\nu} j_\mu$; further $a_\nu^- = \theta$ and $p = 2\pi i \sum_1^k n_\nu j_\nu$ as asserted. In particular, if $k = 1$ we have $j_1 = e$, $p = 2\pi i n_1 e$.

By Theorem 5.12.1, $\mathfrak{B}$ will contain an idempotent $j \neq \theta$, e if and only if it contains an x whose spectrum is not connected. In general the number of linearly independent idempotents is infinite so that the number of linearly independent periods of the exponential function is also infinite. The periods may very well form a non-countable set. As an example consider the closed operator algebra $\mathfrak{B}[I, T]$ generated by a linear bounded self-adjoint operator T on Hilbert space to itself and suppose that the spectrum of T is the interval $\alpha \leqq \lambda \leqq \beta$. Let $E(\lambda)$ be the corresponding resolution of the identity so that $E(\lambda)E(\mu) = E(\lambda)$ for $\alpha \leqq \lambda \leqq \mu \leqq \beta$. Then $E(\lambda)$ is an element of $\mathfrak{B}[I, T]$ and the set of elements $\{2\pi i E(\lambda)\}$, $\alpha \leqq \lambda \leqq \beta$, is a set of periods of the exponential function in this algebra. Cf. section 19.2

References. Dunford [7, 8], Fantappiè [1, 2], Gelfand [4], Lorch [3, 4], Poincaré [1, 2, 3], F. Riesz [2], Taylor [7], Walsh [1].

5. Functions on the Algebra to Scalars

5.19. Linear multiplicative functionals. The last topic to be discussed in this chapter is the theory of bounded linear multiplicative functionals on a (B)-algebra with unit element. It is practically necessary to restrict oneself to the commutative case.

A commutative (B)-algebra $\mathfrak{B}$, being a (B)-space, has an adjoint space $\mathfrak{B}^*$ of linear bounded functionals on $\mathfrak{B}$. We shall be concerned only with that subclass of $\mathfrak{B}^*$ the elements of which are multiplicative. Here the new facts are due to I. Gelfand [4] whose theory of *functions on maximal ideals* will be considered in section 22.18. The equivalent formulation in terms of multiplicative functionals, which we shall consider here, was communicated to the author by N. Dunford.

Definition 5.19.1. *$\mu(x)$ is said to be a bounded linear multiplicative functional on $\mathfrak{B}$ if* (i) $\mu(x) \in \mathfrak{B}^*$ *and* (ii) $\mu(xy) = \mu(x)\mu(y)$ *for all x and y in $\mathfrak{B}$.*

In every (B)-algebra we have the trivial functional $\mu(x) \equiv 0$ which is evidently bounded, linear, and multiplicative. If $\mathfrak{B}$ is the complex field, then $\mu(x) = x$ is the only non-trivial multiplicative functional. In any other (B)-algebra the existence of multiplicative functionals is equivalent to the existence of maximal ideals (see Theorem 22.18.2). In the following we assume the existence of non-trivial multiplicative functionals.

THEOREM 5.19.1. *If $\mu(x)$ is a bounded linear multiplicative functional and if $\mu(x) \not\equiv 0$, then*

(i) $\mu(e) = 1$;

(ii) $\mu(x) \neq 0$ *if x is regular*;

(iii) $\mu(x) \in \sigma(x)$;

(iv) *if $\lambda_0 \in \sigma(x)$, there exists a multiplicative functional such that $\mu(x) = \lambda_0$*;

(v) $\| \mu \| = 1$ *if* $\| e \| = 1$;

(vi) *if x belongs to the domain of definition of the principal extension of $f(\lambda)$, then $\mu[f(x)] = f[\mu(x)]$.*

PROOF. Since $x = ex$ we have $\mu(x) = \mu(e)\mu(x)$ for all x and since $\mu(x) \not\equiv 0$, there exists at least one x_0 such that $\mu(x_0) \neq 0$. Hence $\mu(e) = 1$. This fact also implies $\| \mu \| \geqq 1$ if $\| e \|$ is assumed to be one.

If x is regular, then $\mu(e) = \mu(xx^{-1}) = \mu(x)\mu(x^{-1})$. Thus $\mu(x) \neq 0$ and

$$\mu(x^{-1}) = [\mu(x)]^{-1}. \tag{5.19.1}$$

In particular, it is seen that for a fixed x and $\lambda \in \rho(x)$

$$\mu[R(\lambda; x)] = \frac{1}{\lambda - \mu(x)}. \tag{5.19.2}$$

The left side has a finite value for every λ in $\rho(x)$. This requires that the value of $\mu(x)$ belongs to the spectrum of x.

Conversely, if $\lambda_0 \in \sigma(x)$, then $y = \lambda_0 e - x$ is singular and, by Theorem 22.14.4, there exists a maximal ideal of $\mathfrak{B}$ which contains y. By Theorem 22.18.2 there is a uniquely determined multiplicative functional $\mu(\cdot)$ which vanishes on this maximal ideal and nowhere else. From $\mu(y) = 0$ we get $\mu(x) = \lambda_0$. It should be noted that (iii) and (iv) show that the range of the values of all multiplicative functionals for a fixed x is precisely the spectrum of x.

Since $\sigma(x)$ by Theorem 5.8.2 lies in the circle $| \lambda | \leqq \| x \|$ we have $| \mu(x) | \leqq \| x \|$ and $\| \mu \| \leqq 1$. The opposite inequality was established above and therefore $\| \mu \| = 1$.

Finally, if $f(\lambda)$ is holomorphic in the domain Δ and $x \in \mathfrak{D}(\Delta)$, then $f(x)$ is given by formula (5.17.4). The right member of this formula is the limit of sums of the form

$$f_n(x) = \frac{1}{2\pi i} \sum_{k=1}^{n} f(\lambda_{k,n}) R(\lambda_{k,n}; x)(\lambda_{k,n} - \lambda_{k-1n})$$

and

$$\mu[f_n(x)] = \frac{1}{2\pi i} \sum_{k=1}^{n} f(\lambda_{k,n})[\lambda_{k,n} - \mu(x)]^{-1}(\lambda_{k,n} - \lambda_{k-1n}).$$

Since $\mu(y)$ is a continuous function of y, the left member tends to $\mu[f(x)]$ when $n \rightarrow \infty$, while the right member tends to a Cauchy integral which clearly represents $f[\mu(x)]$. In this argument we have used (5.19.2) and the linearity of the functional. This completes the proof. We refer the reader to section 22.18 for the theory of functions of maximal ideals and the connections with the representation problem for (B)-algebras.

References. Gelfand [4], Hille [11], Šmulian [1].

PART TWO
ANALYTICAL THEORY OF SEMI-GROUPS

Summary. The remainder of this treatise, except for the Appendix, is devoted to the theory of semi-groups with special reference to one-parameter semi-groups of endomorphisms of a (B)-space (see Definition 7.3.6). The theory falls into two sub-divisions, general and special theory. The present Part Two contains the general theory and is divided into ten chapters: *Subadditive Functions*; *Semi-Modules*; *Addition Theorems in a Banach Algebra*; *Semi-Groups in the Strong Topology*; *Laplace Integrals and Binomial Series*; *Generator and Resolvent*; *Generation of Semi-Groups*; *Analytical Semi-Groups*; *Semi-Groups, Ergodic Theory, and Tauberian Theorems*; and *Spectral Theory*.

Chapters VI, VII, and X contain prefatory material: subadditive functions and semi-modules are intimately connected with each other and with the theory of one-parameter semi-groups, and abstract Laplace integrals are indispensible in the discussion of such semi-groups. The foundations of the theory of one-parameter semi-groups are laid in Chapters VIII and IX. We are concerned with a family of endomorphisms $T(\alpha)$, satisfying $T(\alpha)T(\beta) = T(\alpha + \beta)$ for all values α, β of the parameter in an open semi-module of real or complex numbers having $\alpha = 0$ as a limit point. Two entirely different cases arise according as, when $\alpha \to 0$, $T(\alpha)$ tends to the identity in the uniform or in the strong topology. The discussion is not restricted to the functional equation of the exponential function; other addition theorems are also considered as well as the case in which the parameter manifold is a "positive cone" in a (B)-space. The latter includes the case of n-parameter semi-groups.

We return to the one-parameter case in Chapter XI where the infinitesimal generator of the semi-group and its resolvent are discussed. The latter is the Laplace transform of the semi-group operator. The converse problem of constructing a semi-group with given infinitesimal generator is tackled in Chapter XII. The important case in which the semi-group operator is an analytic function of the parameter is studied at length in Chapter XIII. Ergodic theory is in the main a question of the behavior of a semi-group operator $T(\alpha)$ when the parameter tends to zero or infinity; in Chapter XIV we show that ergodic theory is closely related to Tauberian theory of Laplace integrals, applied to the resolvent of the generator of the semi-group. Chapter XV is concerned with spectral theory, operational calculus, approximation of the identity, and boundary value problems for analytical semi-groups.

CHAPTER VI

SUBADDITIVE FUNCTIONS

6.1. Orientation. In section 2.8 we encountered subadditive functionals, that is, functions on an abstract space $\mathfrak{X}$ to E_1 such that

$$f(t_1 + t_2) \leqq f(t_1) + f(t_2), \qquad t_1, t_2 \in \mathfrak{X}.$$

The case in which $\mathfrak{X}$ is a euclidean space is particularly important in analysis. Thus if $\mathfrak{X} = E_n$ and $f(t)$ is positive-homogeneous, we have applications to the theory of convex solids (H. Minkowski [1]) and to the uniqueness theory of differential equations (E. Kamke [1], M. Hukuhara [1]). The special case $n = 1$ is encountered in the theory of moduli of continuity (Ch.-J. de la Vallée-Poussin [1, pp. 7–8]. More recently, A. Beurling [1] and I. Gelfand [6] have considered certain classes of weight factors, the logarithms of which are subadditive functions, with applications to absolutely convergent Fourier integrals and Fourier series, singular integral equations, and Tauberian theorems.

Subadditive functions also play a basic role in the theory of semi-groups where they enter in two different connections. The first instance is in the *theory of additive semi-groups or semi-modules in E_n*. In the simplest and most important case, that in which the semi-module is an open point set having the origin as a limit point, the boundary of the semi-module is defined by a subadditive function in E_{n-1}. The second instance is in the *theory of one-parameter semi-groups $\{T(\alpha)\}$ of endomorphisms of a (B)-space.* Here the parameter set is a semi-module of real or complex numbers and $\log \| T(\alpha) \|$ is a subadditive function of α on this set.

In view of these facts it is necessary for us to include a discussion of subadditive functions and of semi-modules; the former occupies the present chapter, the latter Chapter VII. We restrict ourselves to subadditive functions in E_1 defined on an open semi-module having the origin as a limit point, in other words, one of the intervals $(-\infty, \infty)$, $(-\infty, 0)$ or $(0, \infty)$. This is the most important case for our needs; ultimately the theory has to be carried over to higher dimensions, but this is not so urgent.

It turns out that a finite, measurable subadditive function is necessarily bounded in any closed interval interior to its domain of definition, but may tend to infinity at the end points. It is dominated above by a linear function for large values of t, but there is no universal lower bound, nor is there any such bound for small values of t, if $t = 0$ is an end point. The function need not be continuous anywhere; such properties of continuity and differentiability as it may possess are regulated by its behavior for small values of t.

The presentation is divided into two paragraphs: *Boundedness and Growth*, *Continuity and Differentiability*. The latter includes a discussion of various

associated limit functions which are also subadditive and of moduli of continuity. There are isolated results in the literature, but apparently no systematic discussion of subadditive functions *per se*.

The reader who is anxious to reach the theory of semi-groups as soon as possible can omit material in fine print; the theorems in sections 6.4 and 6.6 are indispensible for the following.

References. Beurling [1], Cooper [1], Gelfand [6], Hardy, Littlewood and Pólya [1], Hille [7, pp. 13, 46–47], Hukuhara [1], Kamke [1], Minkowski [1], Pólya and Szegö [1, p. 17, Ex. 98], and de la Vallée-Poussin [1].

1. Boundedness and Growth

6.2. Preliminaries. In the following we shall be concerned with real functions of a real variable, the domain of definition being one of the intervals I_0 : $(-\infty, \infty)$, I_- : $(-\infty, 0)$, and I_+ : $(0, \infty)$. The symbol I will refer to any one of these intervals. The name "subadditive function" was suggested by M. Riesz.

Definition 6.2.1. *A function $f(t)$ defined on the interval I is said to be subadditive if for all t_1 and t_2 in I we have*

$$f(t_1 + t_2) \leqq f(t_1) + f(t_2). \tag{6.2.1}$$

Theorem 6.2.1. *A positive constant is subadditive in any interval I. If $f_1(t)$ and $f_2(t)$ are subadditive in I and if C_1, C_2 are positive constants, then $C_1f_1(t) + C_2f_2(t)$ is also subadditive in I.*

The simple verification is left to the reader.

Theorem 6.2.2. *If $f(t)$ is subadditive in I, so is $p(t) = \max[0, f(t)]$.*

Proof. The inequality (6.2.1) is certainly satisfied by $p(t)$ for values of t_1 and t_2 such that $f(t_1)f(t_2) \geqq 0$. If instead $f(t_1) > 0$, $f(t_2) < 0$, then

$$p(t_1 + t_2) = \max[0, f(t_1 + t_2)] \leqq f(t_1) = p(t_1) = p(t_1) + p(t_2).$$

It is well known that the functional equation

$$F(t_1 + t_2) = F(t_1) + F(t_2) \tag{6.2.2}$$

has non-measurable solutions in addition to the continuous solution $F(t) = \alpha t$, α arbitrary. Any real solution of (6.2.2) also satisfies (6.2.1), so *there are non-measurable subadditive functions.* Such functions are explicitly excluded from consideration and *all subadditive functions discussed in the following are supposed to be measurable.*

The inequality (6.2.1) is similar to

$$g[\tfrac{1}{2}(t_1 + t_2)] \leqq \tfrac{1}{2}[g(t_1) + g(t_2)] \tag{6.2.3}$$

which characterizes *convex functions*. The two function classes are related, but not very closely. The next two theorems, of which the first is largely due to R. A. Rosenbaum, have a bearing on this situation.

We refer to G. H. Hardy, J. E. Littlewood, and G. Pólya [1] for the properties of convex functions used below. In particular, a measurable convex function is continuous and satisfies

$$g[\alpha t_1 + (1 - \alpha)t_2] \leqq \alpha g(t_1) + (1 - \alpha)g(t_2), \qquad 0 < \alpha < 1. \tag{6.2.4}$$

The function $f(t)$ is *concave* if $-f(t)$ is convex.

THEOREM 6.2.3. (i) *If $f(t)/t$ is decreasing in I_+, then $f(t)$ is subadditive, but need not be convex or concave in I_+.* (ii) *If $f(t)$ is convex and subadditive in I_+, then $f(t)/t$ is decreasing.*

PROOF. (i) We have

$$f(t_1 + t_2) = t_1 \frac{f(t_1 + t_2)}{t_1 + t_2} + t_2 \frac{f(t_1 + t_2)}{t_1 + t_2} \leqq t_1 \frac{f(t_1)}{t_1} + t_2 \frac{f(t_2)}{t_2} = f(t_1) + f(t_2).$$

The function $f(t) = t^{-1} + t^{\frac{1}{2}}$ satisfies the conditions of the theorem, but has a point of inflection at $t = 4$. (ii) Take $0 < a < b$ and put $t_1 = a$, $t_2 = a + b$, $\alpha = a/b$ in (6.2.4). This gives

$$f(b) \leqq \frac{a}{b} f(a) + \left(1 - \frac{a}{b}\right) f(a + b) \leqq \frac{a}{b} f(a) + \left(1 - \frac{a}{b}\right) [f(a) + f(b)]$$

which upon simplification reduces to $af(b) \leqq bf(a)$. This completes the proof.

THEOREM 6.2.4. *A necessary and sufficient condition that a measurable concave function $f(t)$ be subadditive in I_+ is that $f(+0) \geqq 0$.*

PROOF. Theorem 6.4.2 below shows that the condition is necessary. The sufficiency is proved as follows. Since $f(t)$ is concave, $-f(t)$ satisfies (6.2.4) and upon placing $t_1 = 0$, $(1 - \alpha)\, t_2 = a$, $t_2 = b$, we get

$$f(a) \geqq \alpha f(0) + (1 - \alpha) f(b) \geqq \frac{a}{b} f(b).$$

Hence $f(t)/t$ is decreasing so that $f(t)$ is subadditive by the preceding theorem. In the proof we have tacitly assumed $f(t)$ to be continuous to the right at $t = 0$. If this is not true, the desired inequality follows by a suitable passage to the limit. We leave this point to the reader.

In discussing subadditive functions we may disregard the case $I = I_-$ for if $f(t)$ is subadditive in I_-, $f(-t)$ will be subadditive in I_+.

6.3. **Infinitary solutions.** It is desirable for the applications to allow solutions of (6.2.1) which have infinite values, $+\infty$ or $-\infty$. Denoting real numbers by a, addition is defined for these symbols by the conventions: $a + \infty = \infty + \infty = \infty$, $a - \infty = -\infty - \infty = -\infty$. The symbol $\infty - \infty$ is left undefined and if $f(t_1) = +\infty$ while $f(t_2) = -\infty$, then the value

of $f(t_1 + t_2)$ is not restricted at all by these data. A solution of (6.2.1) is said to be *finite in the interval* (a, b) if $f(t) \neq +\infty$ and $-\infty$ when $a < t < b$. *A finite subadditive function is one which is finite in its interval of definition* which is always understood to be open. The basic facts regarding *infinitary solutions* are contained in the following theorem.

THEOREM 6.3.1. *Let $f(t)$ be subadditive in I_0. If $f(a) = -\infty$ for a fixed a, $a > 0$, and $f(t)$ is finite in $(0, a)$, then $f(t) = -\infty$ for $t \geqq a$ and $f(t) = +\infty$ for $t < 0$. If, on the other hand, $f(a) = +\infty$, $a > 0$, then $f(t) = +\infty$ on a subset of $(0, a)$ of measure $\geqq a/2$ and if $f(t)$ is finite for $t > a$, then $f(t) = +\infty$ when $t < 0$.*

PROOF. Suppose that $f(a) = -\infty$. If $t > a$ we may find an h, $0 \leqq h < a$, and a positive integer n such that $t = a + nh$, whence $f(t) \leqq f(a) + nf(h)$ and $f(t) = -\infty$ since $f(h) \neq +\infty$. If t is given, $t < 0$, then there is a quantity b, $a \leqq b$, such that $f(b) = -\infty$ and $0 < t + b \leqq a$. Hence $f(t) \geqq f(t + b) - f(b) = +\infty$.

Suppose instead that $f(a) = +\infty$, $a > 0$. Then if $t_1 > 0$, $t_2 > 0$ and $t_1 + t_2 = a$, we have $+\infty = f(a) \leqq f(t_1) + f(t_2)$, that is, either $f(t_1)$ or $f(t_2)$ is $+\infty$. Since $f(t)$ is measurable by assumption, it follows that $f(t) = +\infty$ on a subset of $(0, a)$ the measure of which is at least $a/2$. If $f(t)$ is finite when $t > a$ and $t_0 < 0$, then $f(t_0) \geqq f(a) - f(a - t_0) = +\infty$.

The following examples show that solutions of the type contemplated in Theorem 6.3.1 really exist:

$$f_1(t) = \begin{cases} +\infty, & t \leqq 0, \\ \cot(\pi t/a), & 0 < t < a, \\ -\infty, & a \leqq t < \infty; \end{cases} \qquad f_2(t) = \begin{cases} +\infty, & t \leqq \dfrac{a}{2} \text{ and } t = a, \\ 0 & \text{elsewhere.} \end{cases}$$

The verification is left to the reader.

6.4. Boundedness.

Finite subadditive functions have remarkable properties of boundedness which will now be investigated.

THEOREM 6.4.1. *If $f(t)$ is subadditive and different from $+\infty$ in I, then $f(t)$ is bounded above in any closed finite interval I^* interior to I. If $f(t)$ is also different from $-\infty$, then $f(t)$ is bounded in I^*.*

PROOF. Suppose first that $I = I_+$, $a > 0$ and $f(a) = A$. For $t_1 + t_2 = a$, $t_1 > 0$, $t_2 > 0$, we have $A = f(a) \leqq f(t_1) + f(t_2)$. It follows that the measure of the set $E_t[f(t) \geqq A/2,\ 0 < t < a]$ is at least $a/2$. Suppose now that $f(t)$ should be unbounded above in some interval (α, β) where $0 < \alpha < \beta < \infty$. We can then find a sequence of points $\{t_n\}$ such that $f(t_n) \geqq 2n$, $t_n \to t_0 \geqq \alpha$. It follows that for every n the set $E_t[f(t) \geqq n,\ 0 < t < \beta]$ has a measure $\geqq \alpha/2$ and hence that $f(t) = +\infty$ in a set of measure $\geqq \alpha/2$. This is a contradiction and shows that to every $\delta > 0$ there is a finite M_δ such that $f(t) < M_\delta$ when $\delta \leqq t \leqq 1/\delta$.

If $I = I_0$ we prove by the same type of argument that $f(t)$ is bounded above also in $-1/\delta \leqq t \leqq -\delta$. But $f(\epsilon) \leqq f(1 + \epsilon) + f(-1)$ and therefore $f(t)$ is bounded above also in $[-\delta, \delta]$ and hence in every interval $[-1/\delta, 1/\delta]$.

Suppose next that $f(t)$ is also different from $-\infty$ in I_+. If $f(t)$ is not bounded below in (α, β) we can find a sequence $\{t_n\}$ such that $f(t_n) \leqq -n$ and $t_n \to t_0$. We set $M = \sup f(t)$ in $(2, 5)$. For any t' in this interval $f(t' + t_n) \leqq$

$f(t') + f(t_n) \leqq M - n$. For large n the intervals $(t_n + 2, t_n + 5)$ contain the fixed interval $(t_0 + 3, t_0 + 4)$ and for every t in this interval $f(t) \leqq M - n$, that is, $f(t) = -\infty$ against the assumption. Thus $f(t)$ is also bounded below in $[\delta, 1/\delta]$ and consequently bounded.

If $I = I_0$ the same argument shows that $f(t)$ is bounded below in $[-1/\delta, -\delta]$. From boundedness in $[1, 3]$ and the inequality $f(\epsilon) \geqq -f(2) + f(2 + \epsilon)$, one infers that $f(t)$ is also bounded below in $[-\delta, \delta]$ and hence bounded in $[-1/\delta, 1/\delta]$. It is possible, however, to prove a sharper result at $t = 0$.

THEOREM 6.4.2. *If $f(t)$ is subadditive in I, then* $\liminf_{t\to 0} f(t)$ *is either* $-\infty$ *or* $\geqq 0$. *In the first case $f(t)$ is infinitary in I.*

PROOF. Put $\liminf_{t\to 0} f(t) = \lambda$ and suppose first that λ is finite. There exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} f(t_n) = \lambda$, $\lim_{n\to\infty} t_n = 0$. For $n \geqq N_\epsilon$ we have then $\lambda - \epsilon \leqq f(2t_n) \leqq 2f(t_n) \leqq 2(\lambda + \epsilon)$ so that $\lambda \leqq 2\lambda$ or $\lambda \geqq 0$. If $\lambda = +\infty$ and $I = I_0$, then $f(t) \equiv +\infty$ in at least one of the subintervals I_+ and I_-. Finally if $\lambda = -\infty$ and $I = I_+$, then we can find a sequence $\{t_n\}$ such that $f(t_n) \leqq -n$ and $t_n \to 0$. From $f(kt_n) \leqq -kn$, one infers that $f(t)$ is unbounded below in every interval (α, β), $0 < \alpha < \beta < \infty$. If $f(t) \not\equiv +\infty$ in I_+, the proof of Theorem 6.3.1 shows that $f(t) \equiv -\infty$. If $I = I_0$ instead, we still conclude that $f(t) \not\equiv +\infty$ in one or both of the subintervals I_+ and I_- implies that $f(t) \equiv -\infty$ in that subinterval. It is possible, however, to have $f(t) = -\infty$ in one subinterval, $+\infty$ in the other, and $f(0)$ perfectly arbitrary.

It should be observed that $\lim_{t\to 0} f(t)$ does not have to exist. An example will be given in section 6.7 of a subadditive function such that $\liminf_{t\to 0} f(t) = 0$, $\limsup_{t\to 0} f(t) = +\infty$.

6.5. Negative subadditive functions. In order to amplify the preceding results we shall consider subadditive functions taking on negative values. The behavior of such functions differs in some respects from that of typical subadditive functions.

THEOREM 6.5.1. *If $f(t)$ is finite and subadditive in I_+ and $f(a) < 0$, $a > 0$, then $f(t) < 0$ for all large positive values of t. If I_+ be replaced by I_0 in the assumption, then in addition $f(t) \geqq 0$ for all negative values of t.*

PROOF. Theorem 6.4.1 shows the existence of a finite M such that $f(t) \leqq M$ when $a \leqq t \leqq 2a$. If now $na \leqq t < (n + 1)a$, then

$$f(t) \leqq f[(n-1)a] + f[t - (n-1)a] \leqq (n-1)f(a) + M \leqq \frac{f(a)}{a}\, t + M,$$

which is negative for all large positive t. Moreover, the inequality shows that $f(t)$ is bounded above by a linear function of t for $t \geqq 2a$. Suppose that $f(t)$ is also defined for $t \leqq 0$ and is subadditive in I_0. Since $f(t)$ is finite and $f(0) \leqq 2f(0)$ we have $f(0) \geqq 0$ and from $0 \leqq f(0) \leqq f(t) + f(-t)$ we conclude that $f(t)$ and $f(-t)$ are not negative simultaneously. In particular, $f(t)$ is certainly positive for all large negative values of t. But if $f(t) < 0$ for any $t < 0$, then the argument used above shows that $f(t) < 0$ for all $t < t_0 < 0$, which is impossible. Hence $f(t) \geqq 0$ when $t \leqq 0$.

We note that if $f(t)$ is subadditive and $f(a) = 0$, then $f(na) \leqq 0$, $n = 1, 2, 3, \cdots$. Here equality may hold for all n as is shown by the example $f(t) = |\sin t|$, $a = \pi$.

The simplest of all functions which are negative and subadditive in I_+ is $f(t) = -t$. By Theorem 6.2.4 any concave decreasing function with $f(+0) = 0$ is negative and subadditive in I_+. The following theorem shows how to construct further such functions, the growth properties of which will be of interest in connection with the discussion in the next section.

THEOREM 6.5.2. *Let $f_0(t)$ be negative and subadditive in I_+ and let $F(t)$ be any positive never decreasing function defined in I_+. Then $f_0(t)\, F(t)$ is negative and subadditive in I_+.*

The simple verification is left to the reader. The theorem shows that it is possible to construct a subadditive function on the interval I_+ which tends faster to $-\infty$ when $t \to \infty$ than any fixed preassigned function of t. This is in marked contrast to the case in which $f(t)$ tends to $+\infty$ with t; here nothing faster than a linear function of t is admissible.

6.6. Rate of growth. A finite subadditive function is bounded in any finite closed interval interior to its interval of definition. It may, however, become unbounded when t approaches either end point of I. We start with a theorem concerning the behavior for large values of t. For the following compare G. Pólya and G. Szegö [1, p. 17, Ex. 98] and the papers by A. Beurling [1], R. Cooper [1], and I. Gelfand [6]. See also Lemma 5.8.1.

THEOREM 6.6.1. *If $f(t)$ is a finite subadditive function on I_0 and if*

$$\inf_{t>0} \frac{f(t)}{t} = \beta \quad \text{and} \quad \sup_{t<0} \frac{f(t)}{t} = \alpha,$$

then

$$\lim_{t\to+\infty} \frac{f(t)}{t} = \beta \quad \text{and} \quad \lim_{t\to-\infty} \frac{f(t)}{t} = \alpha, \tag{6.6.1}$$

$$-\infty < \alpha \leqq \beta < +\infty. \tag{6.6.2}$$

If I_0 is replaced by I_+, then only the first limit has a sense and now $-\infty \leqq \beta < \infty$.

PROOF. Since $f(t)$ is finite, β is either finite or $-\infty$ and α is either finite or $+\infty$. We shall restrict ourselves to proving the existence of the first limit under the assumption that β is finite. The omitted cases follow the same general pattern and will be left to the reader. Choose an a such that $f(a) < (\beta + \epsilon)a$ and let $(n+1)a \leqq t < (n+2)a$, then

$$\beta \leqq \frac{f(t)}{t} \leqq \frac{na}{t}\frac{f(a)}{a} + \frac{f(t-na)}{t} < \frac{na}{t}(\beta + \epsilon) + \frac{f(t-na)}{t}.$$

Letting $t \to +\infty$, the last member tends to $\beta + \epsilon$ since $f(t - na)$ stays bounded by Theorem 6.4.1. It follows that the first limit in (6.6.1) exists and equals β. The inequality $\alpha \leqq \beta$ follows from $0 \leqq f(0) \leqq f(t) + f(-t)$ upon dividing by t and passing to the limit. The other inequalities have already been noted.

The inequality (6.6.2) has an important consequence:

Theorem 6.6.2. *A function $f(t)$, finite and subadditive in I_+ and such that $\beta = -\infty$, does not admit of a subadditive extension to I_0 which is finite for any $t < 0$.*

Proof. Theorem 6.5.2 shows that Theorem 6.6.2 is not vacuous. Suppose now that $f(t)$ is finite and subadditive in I_+, $\beta = -\infty$, and $f(t)$ has been extended as a subadditive function to all of I_0. Theorem 6.5.1 shows that the extension must be non-negative for $t < 0$. Suppose that $f(-b) < +\infty$ so that $f(-kb) < +\infty$ for $k = 1, 2, 3, \cdots$. Any negative t may be written in the form $t = t_0 - kb$ where $b \leqq t_0 < 2b$. Hence we have $0 \leqq f(t) \leqq f(t_0) + f(-kb) \leqq M + kf(-b)$. This shows that the extension is finite for all negative values of t and therefore $f(t)$ is bounded in any finite interval. From the definition of α it follows that $\alpha \geqq -f(-b)/b$ is finite and this contradicts (6.6.2). Hence the only possible extension is $f(t) \equiv +\infty$ for $t < 0$.

Theorem 6.6.1 shows that every finite subadditive function in I_+ is dominated above by a suitably chosen linear function of t for large positive values of t but no such dominant need exist for $-f(t)$ which may grow arbitrarily fast.

A function which is finite and subadditive in I_+ may also become infinite when t decreases to zero. This is shown by

Theorem 6.6.3. *Any function $G(t)$ which is positive and never increasing in I_+ is subadditive in I_+. If $G(t) \to +\infty$ when $t \to 0$, then $G(t)$ does not have a finite subadditive extension for negative values of t. If $f(t)$ is a finite non-negative subadditive function in I_+ so is $f(t)G(t)$.*

Proof. That $G(t)$ and $G(t)f(t)$ are subadditive in I_+ is trivial. Suppose that $G(t)$ is defined and subadditive in I_0 and that $G(t) \to +\infty$ when $t \to +0$. The inequality $G(h) \leqq G(t+h) + G(-t)$, which is valid for every $h > 0$, shows that $G(t) \equiv +\infty$ for $t < 0$.

The results on boundedness and rate of growth may be summarized as follows:

Theorem 6.6.4. *A finite subadditive function is bounded in any finite closed interval interior to its interval of definition I. If $I = I_+$, then $\lim_{t\to+\infty} f(t)/t$ exists and equals $\beta = \inf_{t>0} f(t)/t < +\infty$. A linear function of t dominates $f(t)$ for large t, but if $\beta = -\infty$, $f(t)$ may tend to $-\infty$ faster than any preassigned function when $t \to +\infty$. Further, $\lambda = \liminf_{t\to 0} f(t) \geqq 0$ and $\Lambda = \limsup_{t\to 0} f(t) \leqq +\infty$. If $\Lambda = +\infty$, $f(t)$ may become infinite faster than any preassigned function when $t \to +0$. For the same function $f(t)$, it may happen that $\lambda = 0$, $\Lambda = +\infty$, and $\beta = -\infty$. If either the second or the third relation holds, then $f(t)$ does not admit of a finite subadditive extension in I_-. If $I = I_0$, then $\beta > -\infty$, $\alpha = \lim_{t\to-\infty} f(t)/t = \sup_{t<0} f(t)/t$ exists and $0 \leqq \beta - \alpha < \infty$.*

The behavior of a subadditive function for large values of t is of importance to the applications. See Beurling [1] and Gelfand [6]. The former distinguishes between the non-analytic and the analytic cases according as $\beta - \alpha$ is zero or not. The particular case in which $\alpha = \beta = 0$ and the integral $\int_{-\infty}^{\infty} [f(t)/(1 + t^2)]\, dt$ converges he refers to as the quasi-analytic one. This terminology is a natural one for the problem considered by Beurling.

2. Continuity and Differentiability

6.7. Composition of two-valued subadditive functions. Given the linear functional equation $F(t_1 + t_2) = F(t_1) + F(t_2)$, the assumption that $F(t)$ is finite and measurable ensures that $F(t)$ is bounded, continuous, and differentiable in every finite interval. Much less can be expected if "$=$" be replaced by "$\leqq$". We have seen that a finite measurable subadditive function is bounded in every finite closed interval interior to its interval of definition. It will be shown below that it need not be continuous anywhere, much less differentiable. To bring out this and related facts which will be useful in the discussion, we shall introduce a class of special subadditive functions.

Let Σ be a measurable semi-module of real numbers, that is, $\Sigma = \{\alpha\}$ is a measurable point set and $\alpha, \beta \in \Sigma$ implies $\alpha + \beta \in \Sigma$. Define

$$f(t; \Sigma) = \begin{cases} a, & t \in \Sigma, \\ b, & t \notin \Sigma, \end{cases} \qquad 0 \leqq a \leqq 2b. \tag{6.7.1}$$

This is obviously a subadditive function in I_0. The assumption that Σ is a semi-module may be dropped if in addition $b \leqq 2a$. Incidentally, $f(t; \Sigma)$ *is measurable if and only if* Σ *is; thus there exist bounded non-measurable subadditive functions.*

If now $\{a_n\}$ is any sequence of positive numbers with $\sum a_n$ convergent, if $\{\Sigma_n\}$ is any sequence of distinct measurable semi-modules of real numbers, and if $f(t; \Sigma_n)$ is defined by (6.7.1) with fixed a, b independent of n, then

$$F(t) = \sum_{n=1}^{\infty} a_n f(t; \Sigma_n) \tag{6.7.2}$$

is also subadditive in I_0.

Among these functions we single out the following for special consideration. Let Σ be the set of rational numbers and let $a \neq b$. Then $f(t; \Sigma)$ is a *two-valued measurable subadditive function which is discontinuous for all values of* t. With the aid of this function we can also construct counter examples to Theorem 6.6.4. We choose $a = 0$, $b = 1$ and form $f(t) = t^{-m}f(t; \Sigma) - t^n$ where m and n are arbitrary positive integers. This is obviously a subadditive function in I_+ with $\lambda = 0$, $\Lambda = +\infty$, and $\beta = -\infty$.

Let Σ_n be the set of positive multiples of $1/n$ for $n = 1, 2, 3, \cdots$, let $b = \frac{1}{2}a > 0$, and $a_n = 2^{-n}$. Then

$$F(t) = \sum_{n=1}^{\infty} 2^{-n} f(|t|; \Sigma_n) \tag{6.7.3}$$

is a subadditive function in I_0 which is discontinuous for all rational values of t except $t = 0$ and continuous for irrational t. If n is an integer, $n \neq 0$, then $F(n) = a$ while $\lim_{t\to n} F(t) = \frac{1}{2}a$. Further $\lim_{t\to 0} F(t) = F(0) = \frac{1}{2}a$. This function will serve us as a counter example to Theorem 6.8.2 below. Further counter examples will be constructed with the aid of the same principle in later sections.

6.8. Limit functions and continuity. We now introduce the *upper* and *lower limit functions* $\bar{f}(t)$ and $\underline{f}(t)$ defined as follows:

$$\bar{f}(t) = \lim_{h\to 0} \sup_{|t-u|<h} f(u), \qquad \underline{f}(t) = \lim_{h\to 0} \inf_{|t-u|<h} f(u). \tag{6.8.1}$$

We recall that *$\bar{f}(t)$ is upper semi-continuous* and *$\underline{f}(t)$ lower semi-continuous.*

THEOREM 6.8.1. *If $f(t)$ is subadditive in I so are $\bar{f}(t)$ and $\underline{f}(t)$.*

PROOF. The case of $\underline{f}(t)$ is typical. If h and ϵ are given positive quantities and $\alpha_h = \inf f(u)$ in $(t_1 - h, t_1 + h)$ while $\beta_h = \inf f(u)$ in $(t_2 - h, t_2 + h)$, then there exists a u_1 in the first interval and a u_2 in the second such that $f(u_1) < \alpha_h + \epsilon$, $f(u_2) < \beta_h + \epsilon$. Then $u_1 + u_2$ is a point in $(t_1 + t_2 - 2h, t_1 + t_2 + 2h)$. Hence if γ_{2h} is the infimum of $f(u)$ in this interval,

$$\gamma_{2h} \leqq f(u_1 + u_2) \leqq f(u_1) + f(u_2) < \alpha_h + \beta_h + 2\epsilon.$$

On passing to the limit with h, the inequality

$$\underline{f}(t_1 + t_2) \leqq \underline{f}(t_1) + \underline{f}(t_2) + 2\epsilon$$

results. Since ϵ is arbitrary, it follows that $\underline{f}(t)$ is subadditive in I. The upper limit function $\bar{f}(t)$ is discussed in the same manner.

THEOREM 6.8.2. *If $f(t)$ is subadditive in I_0 and $\omega(t; f) = \bar{f}(t) - \underline{f}(t)$, then*

$$0 \leqq \omega(t; f) \leqq \bar{f}(0). \tag{6.8.2}$$

This inequality is the best of its kind. If $\bar{f}(0) > 0$, then $f(t)$ may be discontinuous everywhere. If $f(t)$ is continuous at $t = 0$ but $f(0) > 0$, the discontinuities of $f(t)$ may still be everywhere dense. If, however, $f(t)$ is continuous at $t = 0$ and $f(0) = 0$, then $f(t)$ is continuous everywhere.

PROOF. Given ϵ and h, $\epsilon > 0$, $h > 0$, and a point t, two points u_1 and u_2 may be found in the interval $(t - h, t + h)$ such that $f(u_1) > \bar{f}(t) - \epsilon$, $f(u_2) < \underline{f}(t) + \epsilon$. Hence

$$\begin{aligned}\omega(t; f) &= \bar{f}(t) - \underline{f}(t) < f(u_1) - f(u_2) + 2\epsilon \\ &\leqq f(u_1 - u_2) + 2\epsilon \leqq \bar{f}(0) + 3\epsilon,\end{aligned}$$

if h is sufficiently small. Since ϵ is arbitrary, (6.8.2) follows.

That this inequality is the best of its kind follows from the examples of the preceding section. For the function $f(|t|; \Sigma)$ where Σ is the set of positive rationals and $a = 0$, $b > 0$ but arbitrary, we have $\omega(t; f) \equiv b = \bar{f}(0; \Sigma)$ and this function is discontinuous everywhere. Formula (6.7.3) exhibits a subadditive function which is continuous at the origin but discontinuous at all other rational points. Here $\bar{F}(0) > 0$ and $\omega(n; F) = \bar{F}(0)$ for integral values of n. Finally, if $f(t)$ is continuous at $t = 0$ and $f(0) = 0$, then $\bar{f}(0) = 0$ and (6.8.2) shows that $f(t)$ is continuous everywhere. This completes the proof. See also Theorem 2.8.1.

Theorem 6.8.2 breaks down if $f(t)$ is defined merely in I_+ since $\bar{f}(0)$ does not exist. The obvious expedient for getting out of this difficulty is to replace $\bar{f}(0)$ by $\Lambda = \limsup_{t\to+0} f(t) = \bar{f}_d(0)$, which is well defined. Though the inequality $\omega(t; f) \leqq \bar{f}_d(0)$ is false, the subadditive inequality will yield information concerning one-sided oscillations and limits.

We introduce the four one-sided limit functions and the corresponding oscillations

$$\bar{f}_l(t) = \lim_{h\to 0} \sup_{t-h<u<t} f(u), \qquad \bar{f}_d(t) = \lim_{h\to 0} \sup_{t<u<t+h} f(u), \tag{6.8.3}$$

$$\underline{f}_l(t) = \lim_{h\to 0} \inf_{t-h<u<t} f(u), \qquad \underline{f}_d(t) = \lim_{h\to 0} \inf_{t<u<t+h} f(u), \tag{6.8.4}$$

$$\omega_l(t; f) = \bar{f}_l(t) - \underline{f}_l(t), \qquad \omega_d(t; f) = \bar{f}_d(t) - \underline{f}_d(t). \tag{6.8.5}$$

We note that if, for instance, $\omega_l(t; f) = 0$, then $\lim_{h\to 0} f(t-h) = f(t-0)$ exists. The following theorem refers to the case $I = I_0$ or I_+. If $I = I_-$ instead we have to make an obvious interchange of left and right in the wording of the theorem.

THEOREM 6.8.3. *If $f(t)$ is subadditive in $I = I_0$ or I_+ so are the one-sided limit functions. Further*

$$0 \leqq \omega_l(t; f) \leqq \bar{f}_d(0), \qquad 0 \leqq \omega_d(t; f) \leqq \bar{f}_d(0) \tag{6.8.6}$$

and these inequalities are the best possible. If $\bar{f}_d(0) > 0$, $f(t+0)$ and $f(t-0)$ need not exist for a single value of t. If $f(+0)$ exists but exceeds zero, $f(t+0)$ may not exist for any $t \neq 0$. If $\bar{f}_d(0) = 0$ so that $f(+0) = 0$, then $f(t+0)$ and $f(t-0)$ exist everywhere and

$$f(t-0) \geqq f(t) \geqq f(t+0), \tag{6.8.7}$$

but $s(t; f) \equiv f(t-0) - f(t+0)$ may be different from zero in an everywhere dense set. Moreover, if $F(t)$ is any positive never-decreasing continuous function tending to $+\infty$ with t, then there exists a subadditive function in I_+ such that $f(+0) = 0$ but $s(t; f) \geqq F(t)$ for infinitely many values of t tending to infinity.

PROOF. The asserted subadditivity is proved as in Theorem 6.8.1. The inequalities (6.8.6) are proved as (6.8.2); we have to observe that the points u_1 and u_2 should be chosen to the same side of t and that $u_1 > u_2$. This is always possible. That (6.8.6) cannot be improved upon is shown by the subadditive function $f(t)$ which is zero or one according as t is rational or not. Here $\omega_l(t; f) = \omega_d(t; f) \equiv 1 = \bar{f}_d(0)$. Further, if Σ is the interval $(0, 1)$ plus the set of rational numbers and if $f(t; \Sigma) = 1$ on Σ and $\frac{1}{2}$ elsewhere, then $f(t; \Sigma)$ and

$$f(t) = \sum_{n=1}^{\infty} 2^{-n} f(nt; \Sigma)$$

are subadditive in I_0. Here $f(+0)$ exists but equals 1 and $f(t+0)$ does not exist for $t \neq 0$ while $f(t-0)$ does not exist for any t.

Suppose now that $\bar{f}_d(0) = 0$ so that $f(+0)$ exists and is 0. It follows from (6.8.6) that $f(t+0)$ and $f(t-0)$ exist everywhere. The inequality (6.8.7) follows from $f(t) \leqq$

$f(t-h)+f(h)$ and $f(t+h) \leqq f(t)+f(h)$ upon letting $h \to 0$. Denoting by $[t]$ the greatest integer $\leqq t$, one verifies easily that $-[t]$ and

$$f(t) = -\sum_{n=1}^{\infty} 2^{-n}[nt]$$

are subadditive; here $f(+0) = 0$ but $s(t; f) > 0$ for all rational values of t. Finally, if $F(t)$ has the stated properties, then by Theorem 6.5.2 we have that $f(t) = -F(t)[t]$ is subadditive in I_+, $f(t) = 0$ in $(0, 1)$ and $s(n; f) = F(n)$ for $n = 1, 2, 3, \cdots$. This completes the proof.

6.9. Continuity in the mean. If $f(t)$ is a finite measurable subadditive function, then $f(t)$ is integrable over any closed interval interior to I. We introduce the mean values

$$(6.9.1) \qquad \begin{aligned} f^*(t) &= \limsup_{h\to 0} \frac{1}{2h}\int_{-h}^{h} f(t+u)\,du, \\ f_*(t) &= \liminf_{h\to 0} \frac{1}{2h}\int_{-h}^{h} f(t+u)\,du, \end{aligned}$$

and set

$$\bar{\omega}(t; f) = f^*(t) - f_*(t).$$

THEOREM 6.9.1. *If $f(t)$ is a finite measurable function which is subadditive in I, then the mean values $f^*(t)$ and $f_*(t)$ are also subadditive in I. If $I = I_0$, then*

$$(6.9.2) \qquad 0 \leqq \bar{\omega}(t; f) \leqq f^*(0).$$

This inequality is the best of its kind; $\bar{\omega}(t; f) = 0$ for almost all t, but if $f^*(0) > 0$ the points where $\bar{\omega}(t; f) > 0$ may be everywhere dense. If $f^*(0) = 0$, $\bar{\omega}(t; f) \equiv 0$ and $f(t)$ is continuous in the mean everywhere.

The proof is obtained by integrating the inequality

$$f(t_1 + t_2 + (\alpha+\beta)s) \leqq f(t_1 + \alpha s) + f(t_2 + \beta s)$$

with respect to s from $-h$ to h and choosing the numbers α and β properly. We omit the details. For the counter examples we use the function $f(t; \Sigma)$ of formula (6.7.1) choosing $a = 1$, $b = \frac{1}{2}$ and Σ in such a manner that (i) Σ is invariant under the translation $s = t + 2$, (ii) the density of Σ is zero at $t = 0$, and (iii) the upper and lower densities are one and zero respectively at $t = 1$. For this function $f^*(0; \Sigma) = \frac{1}{2}$ and $\bar{\omega}(2n+1; f) = \frac{1}{2}$ which shows that (6.9.2) cannot be improved upon. Condensing the singularities in the usual manner, we can obtain a subadditive function which is continuous in the mean at $t = 0$ but at no other rational point. Similar results hold for left- and right-handed mean values.

6.10. Moduli of continuity. An important class of subadditive functions is furnished by the *moduli of continuity* associated with various function spaces. There does not seem to be an adequate discussion of moduli of continuity in the literature so the following considerations may have some independent interest.

Starting with the simplest case, let $C_u(-\infty, \infty)$ be the class of complex-valued functions $f(\xi)$, uniformly continuous in $(-\infty, \infty)$, and form

$$(6.10.1) \qquad \mu(t; f) = \sup_{-\infty<\xi<\infty} |f(\xi+t) - f(\xi)|,$$

$$(6.10.2) \qquad M(t; f) = \max_{0\leqq s\leqq t} \mu(s; f).$$

The name modulus of continuity is usually reserved for $M(t;f)$, but we shall use this term generically for both types of functions as well as for the analogs defined below. When greater precision is necessary, we shall use the term *rectified modulus of continuity* for $M(t;f)$ and its analogs.

In the subspace $C[-\infty, \infty]$ of functions continuous in $-\infty \leqq \xi \leqq \infty$ with the usual norm $\| f \| = \sup_{-\infty<\xi<\infty} |f(\xi)|$, we have

$$\mu(t;f) = \| f(\xi + t) - f(\xi) \| . \tag{6.10.3}$$

These concepts extend to other function spaces. Suppose that the elements of the complex (B)-space $\mathfrak{X}$ are functions $f(\xi)$, defined for $-\infty < \xi < \infty$, and suppose that the translations $f(\xi + t)$, t real but arbitrary, belong to $\mathfrak{X}$ whenever $f(\xi)$ does. Suppose further that the definition of the norm is such that all translations have the same norm, $\| f(\xi + t) \| = \| f \|$, and that the translations are continuous functions of t so that

$$\lim_{t\to 0} \| f(\xi + t) - f(\xi) \| = 0 \text{ when } f \in \mathfrak{X}.$$

We can then define

$$\mu_{\mathfrak{X}}(t;f) = \| f(\xi + t) - f(\xi) \| , \tag{6.10.4}$$

$$M_{\mathfrak{X}}(t;f) = \sup_{0\leqq s\leqq t} \mu_{\mathfrak{X}}(s;f) \tag{6.10.5}$$

as moduli of continuity in $\mathfrak{X}$. The special case in which $\mathfrak{X} = L_p(-\infty, \infty)$, $1 \leqq p < \infty$, is well known. Here all assumptions are satisfied and the corresponding moduli, which will be denoted by $\mu_p(t;f)$ and $M_p(t;f)$, are important concepts in the Lebesgue theory of integration. The main properties of moduli of continuity are listed in

Theorem 6.10.1. *If the space $\mathfrak{X}$ satisfies the conditions stated above, then the moduli $\mu_{\mathfrak{X}}(t;f)$ and $M_{\mathfrak{X}}(t;f)$ are even continuous non-negative subadditive functions in I_0, vanishing for $t = 0$. Further, $M_{\mathfrak{X}}(t;f)$ is never-decreasing for $t > 0$. If $M_{\mathfrak{X}}(t_0;f) = 0$, $t_0 \neq 0$, then $M_{\mathfrak{X}}(t;f) \equiv 0$ and $f(\xi)$ is a constant. The limit of $M_{\mathfrak{X}}(t;f)$ when $t \to \infty$ is $\leqq 2\| f \|$. The functions $\mu(t;f)$ and $M(t;f)$ defined in* (6.10.1) *and* (6.10.2) *have the same properties except that they are bounded if and only if $f(\xi)$ is bounded.*

Proof. It is obvious that both moduli are non-negative and vanish for $t = 0$. Since $\| f(\xi - t) - f(\xi) \| = \| f(\xi) - f(\xi + t) \|$ by the assumed invariance of the norm under translations, the moduli are even functions of t. The same property of invariance plus the triangular inequality for the norm shows that $\mu_{\mathfrak{X}}(t;f)$ is continuous for all t. This makes $M_{\mathfrak{X}}(t;f)$ also continuous. The same properties of the norm give

$$\begin{aligned} \mu_{\mathfrak{X}}(t_1 + t_2;f) &= \| f(\xi + t_1 + t_2) - f(\xi) \| \\ &\leqq \| f(\xi + t_1 + t_2) - f(\xi + t_2) \| + \| f(\xi + t_2) - f(\xi) \| \\ &= \mu_{\mathfrak{X}}(t_1;f) + \mu_{\mathfrak{X}}(t_2;f). \end{aligned}$$

A simple argument shows that $M_{\mathfrak{X}}(t;f)$ is also subadditive, where, if t_1 and t_2 have opposite signs, we have to keep in mind that $M_{\mathfrak{X}}(t;f)$ is even and never-decreasing when $t > 0$.

Since $\mu_{\mathfrak{X}}(t;f)$ is subadditive and never negative, the assumption $\mu_{\mathfrak{X}}(a;f) = 0$ implies that $\mu_{\mathfrak{X}}(t;f)$ is a periodic function of period a. This is still feasible, for instance, in the space $C_u(-\infty, \infty)$, but in a Lebesgue space it would imply $f(\xi) \equiv 0$. In the case of the rectified modulus $M_{\mathfrak{X}}(t;f)$, which is never decreasing for $t > 0$, the assumption $M_{\mathfrak{X}}(a;f) = 0$ implies $M_{\mathfrak{X}}(t;f) \equiv 0$, and this in turn forces $f(\xi)$ to be a constant (or equivalent to a constant). If $f(\xi) \equiv 0$ is the only constant belonging to $\mathfrak{X}$ as in the case $\mathfrak{X} = L_p(-\infty, \infty)$, then we have the stronger conclusion that $M_{\mathfrak{X}}(a;f) = 0$ for an $a \neq 0$ implies $f(\xi) \equiv 0$.

From the obvious inequality $\mu_{\mathfrak{X}}(t;f) \leqq 2\,||f||$, it follows that $\lim_{t\to\infty} \mathrm{M}_{\mathfrak{X}}(t;f) \leqq 2\,||f||$. In some spaces, the numerical factor 2 cannot be replaced by any smaller number; thus in $C[-\infty,\infty]$ the functions $a + b \arctan \xi$ may be used to show that $\lim_{t\to\infty} \mathrm{M}_{\mathfrak{X}}(t;f)/\,||f||$ may have any value between 0 and 2, the limits included. In $L_2(-\infty,\infty)$ the familiar formula

$$\int_{-\infty}^{\infty} |f(\xi + t) - f(\xi)|^2 \,d\xi = 2 \int_{-\infty}^{\infty} [1 - \cos(t\eta)]\, |F(\eta)|^2 \,d\eta,$$

where $F(\eta)$ is the Fourier transform of $f(\xi)$, shows that

$$\lim_{t\to\infty} \mu_2(t;f) = \sqrt{2}\,||f||.$$

This implies that

$$\sqrt{2}\,||f|| \leqq \lim_{t\to\infty} \mathrm{M}_2(t;f) < 2\,||f||. \tag{6.10.6}$$

The discussion of the case in which $f(\xi) \in C_u(-\infty,\infty)$ follows similar lines and is left to the reader. This completes the proof of the theorem.

We turn now to the application of these concepts to the discussion of the continuity properties of subadditive functions.

Theorem 6.10.2. *Let $f(\xi)$ be a continuous subadditive function in I_0 with $f(0) = 0$. Then $f(\xi) \in C_u(-\infty,\infty)$ and*

$$\mu(t;f) = \max\,[f(t), f(-t)]. \tag{6.10.7}$$

In particular, if $f(t)$ is even, then $\mu(t;f) = f(t)$ and if $f(t)$ is also never decreasing for $t > 0$ then $\mathrm{M}(t;f) \equiv f(t)$.

Proof. The inequality

$$-f(-t) \leqq f(\xi + t) - f(\xi) \leqq f(t)$$

shows that

$$|f(\xi + t) - f(\xi)| \leqq \max\,[|f(t)|, |f(-t)|]$$

and this bound is reached either for $\xi = 0$ or for $\xi = -t$. But $0 = f(0) \leqq f(t) + f(-t)$. Hence at least one of the quantities in the last member is non-negative and dominates the absolute value of the other. This proves (6.10.7) and the rest of the theorem is obvious.

Corollary. *If $f(t)$ is a continuous even subadditive function and $f(0) = 0$, then $f(t)$ is the (non-rectified) modulus of continuity of a function in $C_u(-\infty,\infty)$ and if, in addition, $f(t)$ is never decreasing for $t > 0$, then $f(t)$ is a rectified modulus of continuity.*

These conditions plus boundedness are obviously also necessary in order that $f(t)$ be the modulus of continuity of a function in $L_p(-\infty,\infty)$, but we shall not attempt a complete characterization of such moduli here.

6.11. Differentiability. In studying the question of differentiability of subadditive functions we have as usual to start at $t = 0$. The following theorem should be compared with Theorem 6.6.1.

THEOREM 6.11.1. *If $f(t)$ is a finite subadditive function in I_0 and if*

$$\sup_{t>0} \frac{f(t)}{t} = B \quad \textit{and} \quad \inf_{t<0} \frac{f(t)}{t} = A$$

are finite, then

$$\lim_{h\to+0} \frac{f(h)}{h} = B \quad \textit{and} \quad \lim_{h\to-0} \frac{f(h)}{h} = A, \tag{6.11.1}$$

$$A \leqq B. \tag{6.11.2}$$

The same conclusion is valid for $B = +\infty$ and $A = -\infty$ provided $\lim f(h) = 0$ *or* $\liminf f(h) > 0$ *when $h \to 0$ in I_+ and I_- respectively. If I_0 is replaced by I_+ only the first limit has a sense.*

PROOF. We proceed as in Theorem 6.6.1 and discuss only the first limit in detail. It is clear that $-\infty < B$. Suppose that B is finite and choose an a such that $f(a) > (B - \epsilon)a$. Put $a = nh + \delta$, n positive integer, $0 \leqq \delta < h$. Then

$$B - \epsilon \leqq \frac{f(a)}{a} \leqq \frac{f(nh)}{a} + \frac{f(\delta)}{a} \leqq \frac{nh}{a}\frac{f(h)}{h} + \frac{f(\delta)}{a}.$$

Let $h \to 0$; then $nh/a \to 1$ and $f(\delta) \to 0$, whence

$$B - \epsilon \leqq \liminf \frac{f(h)}{h} \leqq \limsup \frac{f(h)}{h} \leqq B$$

so that (6.11.1) holds. The same type of argument holds if $B = +\infty$ and $f(\delta) \to 0$ with δ. On the other hand, if $\liminf f(h) > 0$, we have manifestly $B = +\infty$ and $\lim f(h)/h = +\infty$. The same conclusion, however, is no longer valid if $\liminf f(h) = 0 < \limsup f(h)$ as is seen from the example of the subadditive function which is 0 or 1 according as t is rational or irrational. Finally, formula (6.11.2) is an immediate consequence of the inequality $0 \leqq f(t) + f(-t)$.

We shall now consider the derived numbers of $f(t)$ which will be denoted by a prefixed D with an index $+$ to denote right, $-$ to denote left, used as a superior for upper and as an inferior for lower derived numbers. Thus $D_-f(t)$ is the lower left derived number of $f(t)$.

THEOREM 6.11.2. *If $f(t)$ is finite and subadditive in I_+, then*

$$D^+f(t) \leqq B, \qquad D^-f(t) \leqq B \tag{6.11.3}$$

for all t. In I_- we have instead

$$D_+f(t) \geqq A, \qquad D_-f(t) \geqq A \tag{6.11.4}$$

and if $f(t)$ is finite and subadditive in I_0, all four inequalities hold for all values of t. In the latter case, if A and B are finite, $f(t)$ is necessarily absolutely continuous. In particular, if $A = B$ then $f(t) = At$.

PROOF. The theorem is obviously trivially true if $A = -\infty$ and $B = +\infty$. If B is finite, the two inequalities

$$f(t + h) - f(t) \leqq f(h), \qquad f(t) - f(t - h) \leqq f(h)$$

upon division by h and passage to the limit yield (6.11.3); replacing h by $-h$ and proceeding in the same manner we get (6.11.4). If all four inequalities hold, A and B being finite, the derived numbers are bounded measurable functions. By a classical theorem, due to Lebesgue, $f(t)$ is an indefinite integral of any one of its derived numbers and hence absolutely continuous. Finally if $A = B$ then $f'(t) \equiv A$ and $f(t) = At$ since $f(0) = 0$.

Thus *the conditions, $f(0) = 0, f'(0)$ exists and equals A, single out a unique subadditive function $f(t) = At$.*

6.12. Extensions. The preceding theory admits of extensions in several directions. We note first that the domain of definition of $f(t)$ need not be one of the intervals I_+, I_-, and I_0 considered above. Instead we could allow an arbitrary interval (a, ∞), $a > 0$. We state without proof the following theorems which will be needed later.

THEOREM 6.12.1. *If $f(t)$ is subadditive and finite in (a, ∞) where $a > 0$, then $f(t)$ is bounded above in every interval $(2a + \epsilon, 2a + 1/\epsilon)$ and bounded below in $(a, a + 1/\epsilon)$. If $f(t)$ is unbounded in $(a, 2a + \epsilon)$, then it does not admit of a finite subadditive extension for $t < a$.*

EXAMPLE. $f(t) = a^2/(3\,at - t^2 - 2a^2)$ for $a < t < 2a$ and $f(t) = 1$ for $2a \leqq t$ is subadditive in (a, ∞) and not bounded above in $(a, 2a)$ though it is finite.

THEOREM 6.12.2. *Under the assumptions of the preceding theorem $\lim_{t\to\infty} f(t)/t$ exists and equals* $\inf f(t)/t$ *for $t > a$.*

We shall not discuss questions of continuity of such subadditive functions. If the function does not admit of a subadditive extension down to zero we loose our method of attack on questions of continuity and differentiability and there is no obvious alternate method available.

We can also consider subadditive functions of n variables. We have then to discuss functions $f(\mathrm{t})$ on E_n to E_1 satisfying

$$f(\mathrm{t}_1 + \mathrm{t}_2) \leqq f(\mathrm{t}_1) + f(\mathrm{t}_2) \tag{6.12.1}$$

for all t_1, t_2 in the domain of definition, Σ_n say. Here $\mathrm{t}_1 + \mathrm{t}_2$ is the vector sum of t_1 and t_2, that is, the vector whose components are the sums of corresponding components of t_1 and t_2. The set Σ_n should obviously be a semi-module: t_1, $\mathrm{t}_2 \in \Sigma_n$ should imply $\mathrm{t}_1 + \mathrm{t}_2 \in \Sigma_n$. Actually the relationship between semi-modules and subadditive functions is mutual; as we shall see in the next chapter, if Σ_n is open and has the origin as limit point then the boundary of Σ_n is determined by a subadditive function in $(n - 1)$ variables.

The inequality (6.12.1) was used by Minkowski [1] to characterize convex solids. Any non-negative continuous function $f(\mathrm{t})$ satisfying (6.12.1) and such that $f(\alpha \mathrm{t}) = \alpha f(\mathrm{t})$ for all t and all positive α defines a convex solid containing the origin. If also $f(-\mathrm{t}) = f(\mathrm{t})$, then the origin is the center of the solid.

The same class of functions arises in the uniqueness theory of differential equations (generalization of the Lipschitz condition, see E. Kamke [1]). From this point of departure M. Hukuhara [1] has determined the structure of all positive-homogeneous continuous subadditive functions in E_n.

In these investigations $\Sigma_n = E_n$ and the assumption of positive-homogeneity simplifies the discussion very much. This assumption may be dropped, however, and the analysis may be based upon the same simple tools as in the case of one dimension. It is of course to be expected that some new features will arise. Thus, if $\Sigma_n = E_n$ and $f(\mathbf{t})$ satisfies a suitable measurability and finiteness condition, then Theorem 6.6.1 generalizes to an assertion that

$$\lim_{\rho\to\infty} \rho^{-1}f(\rho\mathbf{u}) = g(\mathbf{u}), \qquad \|\,\mathbf{u}\,\| = 1, \tag{6.12.2}$$

exists and is bounded. If $n = 2$, then $g(\mathbf{u})$ is the function of support of a bounded convex region. See section 13.3 where a similar investigation is carried out for an arbitrary angular semi-module Σ_2. Subadditive functions in n dimensions are investigated in detail in a forthcoming paper by R. A. Rosenbaum.

CHAPTER VII

SEMI-MODULES

7.1. Orientation. The present chapter serves several different purposes. It contains the basic definitions in the theory of abstract semi-groups and it gives the elements of a theory of semi-modules (= additive abelian semi-groups) with special reference to semi-modules in a euclidean space. Such semi-modules of real or complex numbers form the parameter manifolds of one-parameter semi-groups of linear bounded transformations, the study of which will start in the next chapter and occupy the greater part of this treatise. This fact justifies our studying the one- and two-dimensional cases at some length.

There is a peculiar relationship between semi-modules and subadditive functions which was discovered by Max Zorn in 1942 and the present chapter is largely based upon Zorn's work. In the case of an angular semi-module (= open, additive set whose closure contains the origin) the boundary is determined by a subadditive function whose domain of definition is an angular semi-module of next lower dimension.

There are three paragraphs: *Semi-Groups*, *Semi-Modules in E_n*, and *Topological Semi-Modules*. References are found at the end of the first two paragraphs.

1. Semi-Groups

7.2. Abstract semi-groups. The notion of a *semi-group* is of much more recent origin than that of a group. It seems to have made its first appearance in the literature in 1904 in the treatise of J. A. de Séguier on the theory of abstract groups [1, p. 8] which was followed a year later by a paper by L. E. Dickson [1] devoted to the subject. See also G. Frobenius and I. Schur [1]. During the last fifteen years there have been sporadic papers on the algebraic theory of semi-groups; an incomplete list of such papers is to be found in the References at the end of this paragraph. However, the main importance of the semi-group concept does not seem to lie in the algebraic field, but rather in the applications to analysis where topological semi-groups and in particular one-parameter semi-groups of linear transformations on a function space to itself come up in the most diversified connections. For such semi-groups, topological and analytical methods are available to complement the algebraic ones and a much richer theory results. We start the discussion by giving the formal definition of an abstract semi-group.

DEFINITION 7.2.1. *An abstract semi-group* $\mathfrak{S}$ *is a system of elements which may be combined by a single-valued, binary, and associative operation under which* $\mathfrak{S}$ *is closed. Thus*

(i) *to every pair of distinct or equal elements* a *and* b *of* $\mathfrak{S}$, *taken in this order, there is a unique element* $a \circ b \in \mathfrak{S}$, *and*

(ii) $a \circ (b \circ c) = (a \circ b) \circ c$.

The reader will recognize (i) and (ii) as two of the classical postulates for abstract groups. He will note that *the existence of a unit element and of inverses is not assumed.* The semi-groups considered in this treatise will usually have a unit element, however, and some elements may have inverses. When the notion of a semi-group was first introduced by de Séguier and Dickson, a *law of cancellation* was also assumed:

(iii) *if for any three elements* a, b, c *either* $a \circ b = a \circ c$ *or* $b \circ a = c \circ a$, *then* $b = c$.

If the semi-group has a finite number of elements, (iii) implies that $\mathfrak{S}$ is a group which is not the case if merely (i) and (ii) are assumed. We shall not assume (iii) anywhere in this treatise. On the other hand we shall frequently assume:

(iv) *the operation is commutative,* $a \circ b = b \circ a$.

DEFINITION 7.2.2. *A homomorphism of a semi-group* $\mathfrak{S}$ *onto a semi-group* $\mathfrak{S}'$ *is a single-valued transformation* $x \to x'$ *mapping* $\mathfrak{S}$ *onto all of* $\mathfrak{S}'$, *and such that* $(x \circ y)' = x' \circ y'$ *for all* x, y *in* $\mathfrak{S}$. *A homomorphism which is one-to-one is called an isomorphism. An isomorphism of a semi-group onto itself is called an automorphism.*

DEFINITION 7.2.3. $\mathfrak{S}$ *is a topological (Hausdorff, metric) semi-group if* $\mathfrak{S}$ *is a topological (Hausdorff, metric) space in addition to being an abstract semi-group and if the operation satisfies condition* MT (MH, MD) *of Definition* 1.13.1. *In particular, if the metric is defined in terms of a norm subject to the condition* $\| a \circ b \| \leqq \| a \| \, \| b \|$, *we speak of a normed semi-group.*

7.3. Transformation semi-groups. An abstract semi-group is usually obtained from a transformation semi-group by a process of abstraction which disregards the nature of the elements and preserves only their mode of combination. Conversely, an abstract semi-group may be realized as a transformation semi-group. The following definitions explain the terminology.

DEFINITION 7.3.1. *A set* $\mathfrak{T}$ *of transformations* T_α *on an abstract space* $\mathfrak{X}$ *to itself is a transformation semi-group if* $T_\alpha T_\beta \in \mathfrak{T}$ *whenever* T_α *and* $T_\beta \in \mathfrak{T}$.

DEFINITION 7.3.2. *A transformation semi-group* $\mathfrak{T}$ *is a realization of the abstract group* $\mathfrak{S}$ *if to every element* a *of* $\mathfrak{S}$ *corresponds an element* $T(a)$ *of* $\mathfrak{T}$ *in such a manner that* $T(a \circ b) = T(a)\, T(b)$. *A realization is faithful if* $a \neq b$ *implies* $T(a) \neq T(b)$.

Thus a realization is a homomorphic mapping of the abstract semi-group onto a semi-group of transformations. Every abstract semi-group admits of a realization by means of *left-translations*. We take for $\mathfrak{X}$ the semi-group manifold itself and define $T(a)$ to be the transformation $y = a \circ x$. Then there is a unique $T(a)$ to every a in $\mathfrak{S}$ and $T(a \circ b) = T(a)\ T(b)$. The realization is certainly faithful if $\mathfrak{S}$ has a unit element or, more generally, if $a \circ x = b \circ x$ for all x in $\mathfrak{S}$ implies $a = b$.

This method does not always lead to a faithful realization, however. Thus in the semi-group of matrices of the form

$$\begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix},$$

two left-translations $T(a)$ and $T(b)$ always coincide if $a_{11} = b_{11}$ regardless of the values of a_{12} and b_{12}.

Definition 7.3.3. *A realization $\mathfrak{T}$ of $\mathfrak{S}$ is called a representation of $\mathfrak{S}$ if $\mathfrak{X}$ is a Banach space and the elements of $\mathfrak{T}$ are linear bounded transformations. If $\mathfrak{X}$ is n-dimensional, the representation is of degree n.*

In classical algebra it is customary to restrict oneself to representations of finite degree. This would not be appropriate for the applications which we have in view; even if representations of finite degree exist, it is those of infinite degree which are apt to be of interest to the analyst. Thus practically all the transformation semi-groups which are studied in Parts Two and Three of this book are representations of infinite degree of very simple abstract semi-groups for which one can find trivial faithful representations of degree one. The next definitions serve to introduce these special semi-groups.

Definition 7.3.4. *A semi-module is an additive abelian semi-group, that is, $x \circ y = y \circ x = x + y$ for $x, y \in \mathfrak{S}$.*

Definition 7.3.5. *A one-parameter semi-group $\mathfrak{S}$ is the homomorphic image of a semi-module Σ of real or complex numbers. Σ is called the parameter manifold of $\mathfrak{S}$.*

Thus $\mathfrak{S} = \{x_\alpha\}$ is a one-parameter semi-group if the law of composition reads

$$x_\alpha \circ x_\beta = x_\beta \circ x_\alpha = x_{\alpha+\beta}, \qquad \alpha, \beta \in \Sigma. \tag{7.3.1}$$

Such a semi-group has a faithful trivial representation by a semi-group of translations:

$$T(\alpha)x = x + \alpha,$$

where $T(\alpha)$ operates on the space of real or of complex numbers. This representation is of degree one. It is a special case of the representations defined in

DEFINITION 7.3.6. *Given a (B)-space $\mathfrak{X}$ and a family $\mathfrak{T} = \{T(\alpha)\}$ of endomorphisms of $\mathfrak{X}$ (= linear bounded transformations on $\mathfrak{X}$ to itself), where $T(\alpha)$ is defined for α in a semi-module Σ of real or complex numbers. $\mathfrak{T}$ is called a one-parameter semi-group of endomorphisms if for all α, β in Σ we have*

$$T(\alpha)T(\beta) = T(\alpha + \beta). \tag{7.3.2}$$

The study of such semi-groups will be our main concern in the remainder of this book.

7.4. Some examples of semi-groups. We list some simple illustrations of the semi-group concept.

(1) A ring is a semi-group under addition as well as under multiplication. A group is also a semi-group.

(2) The positive real numbers form a semi-group both under addition and under multiplication; the negative numbers only under addition.

(3) The complex numbers inside the unit circle form a semi-group under multiplication.

(4) The complex numbers z in the sector $\alpha < \arg z < \beta$ form a semi-module if $\beta - \alpha \leqq \pi$.

(5) The complex numbers $z = x + iy$ such that $x > |y|^\alpha$, α fixed, $0 < \alpha \leqq 1$, form a semi-module.

The following transformations give examples of one-parameter semi-groups of linear bounded transformations.

(6) Let B be the class of analytic functions $f(z)$, bounded and holomorphic for $\Re(z) > 0$, with $\|f\| = \sup |f(z)|$. Let a linear bounded transformation T_α on B to itself be defined by $T_\alpha[f] = f(z + \alpha)$ when $\Re(\alpha) > 0$. Then $\mathfrak{T} = \{T_\alpha\}$ is a one-parameter semi-group.

(7) Let $L(0, 1)$ be the class of integrable functions on the interval $(0, 1)$ with $\|f\| = \int_0^1 |f(t)|\, dt$. For $\Re(\alpha) > 0$ define

$$T_\alpha[f] = \frac{1}{\Gamma(\alpha)} \int_0^t (t - u)^{\alpha-1} f(u)\, du.$$

Then $\mathfrak{T} = \{T_\alpha\}$ is a one-parameter semi-group of linear bounded transformations on $L(0, 1)$ to itself.

In (6) and (7) the parameter manifold is the semi-module $\Re(\alpha) > 0$. In both cases the identity transformation can be adjoined and serves as the unit element but inverses do not exist or, to be more precise, the inverse of T_α exists as an unbounded transformation with domain dense in the space in question. In the following example the parameter manifold is the positive real axis which is a semi-module by (2).

(8) Let $C[0, \infty]$ be the (B)-space of functions $f(t)$ continuous for $0 \leqq t \leqq \infty$ and with $\|f\| = \sup |f(t)|$. For $\alpha > 0$ define $T_\alpha[f] = f(t + \alpha)$. Then $\mathfrak{T} = \{T_\alpha\}$ is a one-parameter semi-group of linear bounded transformations on $C[0, \infty]$ to itself.

References (algebraic theory of semi-groups).
History: Dickson [1], Frobenius and Schur [1], de Séguier [1].
Arithmetic: Arnold [1], Lorenzen [1].
Representation: Clifford [1], Suschkewitsch [2].
Structure: Dubreil [1], Rees [1], Suschkewitsch [1].

2. Semi-Modules in E_n.

7.5. Topological considerations. We shall be concerned with the simplest of all infinite semi-groups: *the semi-modules in an n-dimensional euclidean space* E_n. For a given n the set of distinct semi-modules in E_n is non-denumerable and the classification and study of the structure of the various types is essentially a topological problem which, as far as we know, is unsolved even for $n = 1$.

In the case of modules the situation is different (see J. Nielsen [1] and V. Bergström [1]). The whole space is the only open module. If the module is a closed set, then it is the direct sum of one-dimensional modules which are either equivalent to the module of real numbers or else to a module generated by a single element. The general case requires further analysis.

We shall not undertake a general study of the structure of n-dimensional semi-modules, but we shall introduce some concepts which would seem to be useful in such a study and which will be needed later, and we shall solve the structure problem for a particular case which is basic for the theory of one-parameter semi-groups.

Definition 7.5.1. *An element of a semi-module is called reducible if it is the sum of non-zero elements of the semi-module, otherwise irreducible. The set of all irreducible elements is the irreducible core. If the latter is vacuous, the semi-module is called indefinitely reducible.*

Definition 7.5.2. *If S is an arbitrary set of vectors in E_n, the least semi-module of E_n which contains S is called the additive resultant of S and is denoted by $(S)_a$.*

The semi-modules in examples (4) and (5) of section 7.4 are indefinitely reducible. The semi-module of real numbers $t > a > 0$ has the interval $a < t \leqq 2a$ as its irreducible core and is the additive resultant of its core. These examples suggest that a general semi-module may be the direct sum of the additive resultant of its irreducible core and an indefinitely reducible semi-module.

Theorem 7.5.1. *The additive resultant of S is uniquely determined as the intersection of all semi-modules in E_n containing S and $(S)_a$ consists of precisely those vectors of E_n which are obtainable as finite sums of vectors in S.*

The proof is immediate.

We shall need some properties of the additive resultant in later applications. It is clear that $(S)_a$ *is open (closed) whenever S is open (closed). If S is open and contains the origin, then $(S)_a$ is the whole space.* The connectivity questions are less obvious. Since the additive resultant of an open sphere has only a finite number of components, the additive resultant of an open connected set has the same property.

The situation is somewhat simpler in the case of convex sets. In this case we have still another representation of the additive resultant. Let S_α denote the image of S under the affine transformation $x' = \alpha x$, $\alpha > 0$. Let $x \in (S)_a$. We can then find an integer ν and ν vectors $x_1, x_2, \cdots, x_\nu \in S$ such that $x =$

$x_1 + x_2 + \cdots + x_\nu$. Since S is convex $y_\nu = \nu^{-1}(x_1 + \cdots + x_\nu) \in S$ and $x = \nu y_\nu$, that is, $x \in S_\nu$. Conversely $S_\nu \subset (S)_a$ for all ν so that

$$(7.5.1) \qquad (S)_a = \bigcup_\nu S_\nu, \qquad S_1 = S \text{ convex.}$$

LEMMA 7.5.1. *If S is a convex open set in E_n, then $(S)_a$ is connected if and only if $S_1 \cap S_2 \neq \emptyset$ and if this condition is satisfied, $S_\nu \cap S_{\nu+1} \neq \emptyset$ for all ν.*

PROOF. Let x be a point in $S_1 \cap S_2$. Since it is in S_2, there is a point y in S_1 such that $x = 2y$. Since x is also in S_1, the whole segment xy is in S_1. If ν is a positive integer, $\nu > 1$, $z_\nu = (1 - \nu^{-1})y + \nu^{-1}x$ is a point of this segment and νz_ν is in S_ν. But a simple calculation shows that $\nu z_\nu = (\nu + 1)y$, which is in $S_{\nu+1}$. Hence $S_\nu \cap S_{\nu+1} \neq \emptyset$ for all ν. This implies that $X_\nu = \bigcup_{\mu=1}^{\nu} S_\mu$ is connected for all ν and, consequently, that $(S)_a$ has the same property. This shows that the condition is sufficient. But if $S_1 \cap S_2 = \emptyset$, then the equation $x = 2y$ cannot be satisfied by x and y in S_1. If x and y are points in S_1 on the same ray (half-line) from the origin, we have consequently always $\| x \| < 2 \| y \|$. A fortiori, $\| x \| < \nu \| y \|$ for $\nu > 2$, so that $S_1 \cap S_\nu = \emptyset$ for $\nu \geqq 2$ and $(S)_a$ is not connected so the condition is also necessary.

At this juncture a stronger result suggests itself, viz., that $(S)_a$ is actually simply-connected when it is connected, S being an open convex set. This is trivially true in the linear case, but far from obvious in the plane or in E_n. The conjecture has been proved by E. G. Begle whose proof we reproduce below with his permission.

THEOREM 7.5.2. *The additive resultant of an open convex set in E_n is simply-connected whenever it is connected.*

PROOF. The case in which the origin is in the set S can be dismissed as trivial since $(S)_a$ is then the whole space. Next, let the origin be a boundary point of S and $x \in S$. Then all points αx with $0 < \alpha < 1$ are in S and in $(S)_a$ if $0 < \alpha < \infty$. This makes $(S)_a$ trivially connected. If y and z are two points of $(S)_a$, then there exists positive numbers α and β and two points u and v in S such that $y = \alpha u$, $z = \beta v$. Since the line segment joining u and v is in S, the line segment joining y and z is in $(S)_a$. Thus $(S)_a$ is a convex open set and $(S)_a$ is simply-connected.

Leaving these trivial cases to one side, we assume that S has a positive distance from the origin and that $S_1 \cap S_2 \neq \emptyset$ so that $(S)_a$ is connected. We may also assume that S is bounded, since the unbounded case can be handled by a suitable limiting process once it is known that the theorem is true for bounded convex sets.

It is sufficient to show that $X_p = \bigcup_1^p S_j$ is simply-connected for all p. For, on one hand, we observe that every closed curve of $(S)_a$ is a compact set, and hence lies inside some sphere in E_n with center at the origin. On the other hand, $S = S_1$ is at a positive distance from the origin, and therefore for sufficiently large k, S_k lies outside any given sphere.

To continue the proof, we require the following

LEMMA 7.5.2. *For any j, k, $j < k$, we have*

$$S_j \cap S_k = S_j \cap S_{j+1} \cap \cdots \cap S_{k-1} \cap S_k . \tag{7.5.2}$$

PROOF. It is sufficient to show that the left side of (7.5.2) is a subset of the right side, since the opposite inclusion is obvious. Let z be any point of $S_j \cap S_k$. Then $z = jx = ky$ where x and y are both in S_1. Then the segment $[y, x]$ is also in S_1. Consider now the segments $[jy, jx]$, $[(j + 1)y, (j + 1)x]$, $\cdots$, $[(k - 1)y, (k - 1)x]$, $[ky, kx]$. The first of these is in S_j, the second in S_{j+1}, and so on. We assert that these segments all contain the point z. The first point of each segment is $\leqq ky = z$ with obvious interpretation of the inequality. The end point of each segment is $\geqq jx = z$ which proves our assertion.

The main theorem now follows by induction, as follows:

Certainly $X_1 = S_1$ is simply-connected. Let us assume that X_p is simply-connected. Then $X_{p+1} = X_p \cup S_{p+1}$. Now $X_p \cap S_{p+1} = \bigcup_{j=1}^{p} (S_j \cap S_{p+1})$. But, by the lemma above, $S_j \cap S_{p+1} = S_j \cap S_{j+1} \cap \cdots \cap S_p \cap S_{p+1}$. Hence

$$X_p \cap S_{p+1} = \bigcup_{j=1}^{p}(S_j \cap S_{j+1} \cap \cdots \cap S_p \cap S_{p+1}) \subset S_p \cap S_{p+1} .$$

But S_p and S_{p+1} are both convex, so $S_p \cap S_{p+1}$ is also convex and hence connected. Since S_{p+1}, being convex, is simply-connected, and X_p, by hypothesis, is also simply-connected, we can now apply the well-known theorem that if the intersection of two simply-connected sets is connected, their union is simply-connected. This proves the theorem. We observe that the same proof applies if S is closed.

7.6. Angular semi-modules. As a general discussion of the structure problem for semi-modules would take us too far afield, we shall impose restrictions which single out a well defined class of semi-modules with comparatively simple properties. For the following discussion see E. Hille and M. Zorn [1].

DEFINITION 7.6.1. *A semi-module in a topological additive group is called an angular semi-module if it is an open point set having the origin as a limit point.*

We shall usually assume that the semi-module is a subset of E_n and a detailed discussion will be carried through only for $n = 2$.

We note a striking difference between modules and semi-modules: the only module in E_n satisfying the conditions of this definition is the space itself. In one dimension there are only two proper semi-modules: the positive axis and the negative axis which of course are equivalent under a reflection. In two dimensions we have a much greater variety, but the various types can be characterized in simple terms by means of subadditive functions defined on one-dimensional semi-modules. This characterization extends to higher dimensions: the angular

semi-modules in E_n are determined by subadditive functions defined on angular semi-modules in E_{n-1}. The relationship between angular semi-modules and subadditive functions is consequently of recursive nature.

We start the discussion by proving some results of varying degree of generality which are needed for our problem.

We observe first that *in any Hausdorff additive group, the closure of a semi-module is itself a semi-module.*

THEOREM 7.6.1. *If Σ is a semi-module in E_n and if every sphere with center at the origin contains an element of Σ different from* 0, *then there exists a vector $b \neq 0$ such that the ray ρb, $\rho \geqq 0$, is in the closure of Σ.*

PROOF. We denote the length of the vector v by $\| v \|$. By assumption there is a sequence $\{a_j\}$, $a_j \in \Sigma$, $a_j \neq 0$ with $\lim a_j = 0$. The unit vectors $a_j/\| a_j \|$ must have at least one limit point in E_n and without loss of generality we may assume $\lim a_j/\| a_j \| = b$ where $\| b \| = 1$. If ρ is given, $\rho \geqq 0$, we set $n_j = [\rho/\| a_j \|] + 1$ where $[\alpha]$ is the largest integer $\leqq \alpha$. Then we have $\lim n_j \| a_j \| = \rho$ and the relation

$$\rho b = (\lim n_j \| a_j \|)(\lim a_j/\| a_j \|) = \lim n_j a_j$$

shows that ρb is the limit of a sequence from Σ.

THEOREM 7.6.2. *If $\mathfrak{X}$ is a Hausdorff additive group in the sense of Definition* 1.8.1 (2) *and if $\mathfrak{S}$ is an angular semi-module in $\mathfrak{X}$, then $\mathfrak{S}$ is the interior of its own closure.*

PROOF. We have to prove that if a point x of $\mathfrak{X}$ does not belong to $\mathfrak{S}$, then every neighborhood U_x of x contains an open set which is not void and has no points in common with $\mathfrak{S}$. To prove this we take a neighborhood U_0 of the zero element such that $x - U_0$ is in U_x; in this neighborhood there will be a vector y which, together with a full neighborhood U_y, is contained in $\mathfrak{S} \cap U_0$. The non-void open set $x - U_y$ is contained in U_x but has no points in $\mathfrak{S}$; for if $x - u$ were in $\mathfrak{S}$, $x = u + (x - u)$ would be, which is not true. This proves the theorem. An important consequence is

THEOREM 7.6.3. *Under the assumptions of the preceding theorem $\mathfrak{S} + \bar{\mathfrak{S}} \subset \mathfrak{S}$.*

PROOF. It is required to show that if x and y are arbitrary elements of $\mathfrak{S}$ and $\bar{\mathfrak{S}}$ respectively, then their sum belongs to $\mathfrak{S}$. Since the closure of a semi-group is a semi-group, $\mathfrak{S} + \bar{\mathfrak{S}}$ is first of all contained in $\bar{\mathfrak{S}}$ so that $x + y$ is in $\bar{\mathfrak{S}}$. But with $x + y$ there is a full neighborhood of the form $U_x + y$ also in $\bar{\mathfrak{S}}$; in other words, every point of $\mathfrak{S} + \bar{\mathfrak{S}}$ is an interior point of $\bar{\mathfrak{S}}$ and therefore contained in $\mathfrak{S}$ itself.

THEOREM 7.6.4. *If Σ is an angular semi-module in E_n, there exists at least one vector $b \neq 0$ such that $x \in \Sigma$ implies $x + \rho b \in \Sigma$ for all $\rho \geqq 0$.*

PROOF. It suffices to choose the vector b in $\bar{\Sigma}$ which is furnished by Theorem 7.6.1 and then to apply Theorem 7.6.3. Actually a slightly sharper statement could be made: to every $x \in \Sigma$ there is a positive $\epsilon(x)$ such that $x + \rho b \in \Sigma$ for $\rho > -\epsilon(x)$.

All that we have said so far in this section applies in particular to an angular semi-module in E_2. We shall now concentrate on this case and determine the structure of Σ. Since the properties of Σ are invariant under rotations about the origin, we may take b as the vector $(1, 0)$ and identify E_2 with the complex plane. All vectors in E_2 are then of the form $z = x + iy$ and if $x_0 + iy_0 \in \Sigma$, then $x + iy_0 \in \Sigma$ for $x > x_0 - \epsilon(z_0)$. Now it is obvious that the characteristic properties of an angular semi-module (additive, open point set whose closure contains the zero element) are preserved under a projection on a linear subspace. Hence we have

THEOREM 7.6.5. *The projection of Σ on the imaginary axis is one of the three sets $E(y > 0)$, $E(y < 0)$, and $E(-\infty < y < \infty)$.*

It is clear that the first two alternatives are equivalent under a reflection of Σ in the real axis. We may consequently restrict ourselves to the case in which the projection of Σ is one of the sets $E(y > 0)$ or $E(-\infty < y < \infty)$. We denote the projection by Π. The last two theorems show that we may introduce a function $f(y)$ defined as a real number or $-\infty$ for $y \in \Pi$ by

$$f(y) = \liminf x, \qquad x + iy \in \Sigma. \tag{7.6.1}$$

We have then the basic

THEOREM 7.6.6. *$f(y)$ is a subadditive, upper semi-continuous function on Π such that $\liminf_{y\to 0} f(y) = 0$ or $-\infty$. In the latter case, $f(y) \equiv -\infty$ in Π.*

PROOF. Suppose that $x_1 + iy_1$ and $x_2 + iy_2$ are two points of Σ. This implies that $f(y_1) + \delta + iy_1$ and $f(y_2) + \delta + iy_2$ are in Σ for every $\delta > 0$, hence $f(y_1) + f(y_2) + 2\delta + i(y_1 + y_2)$ is in Σ so that $f(y_1 + y_2) < f(y_1) + f(y_2) + 2\delta$. Since δ is arbitrary, $f(y)$ is subadditive. The fact that Σ is open implies that to every y_0 in Π there is a $\delta > 0$ such that all points $x + iy$ with $y_0 - \delta < y < y_0 + \delta$, $x > f(y_0) + \delta$ belong to Σ. This implies that $f(y) < f(y_0) + \delta$ for $|y - y_0| < \delta$ so that $f(y)$ is upper semi-continuous. Finally, $z = 0$ is supposedly a limit point of Σ. This requires that $\liminf_{y\to 0} f(y) \leqq 0$. As an upper semi-continuous function, $f(y)$ is measurable so that Theorem 6.4.2 applies. This theorem gives two alternatives: either $\liminf_{y\to 0} f(y) \geqq 0$ or it is $-\infty$. In the former case we see that $\liminf_{y\to 0} f(y) = 0$. In the latter case we have $f(y) \equiv -\infty$ since $f(y) \not\equiv +\infty$ for every y in Π. This completes the proof. If $f(y) \equiv -\infty$, then obviously Σ is either the upper half-plane $y > 0$ or the whole plane according as Π is $E(y > 0)$ or $E(-\infty < y < \infty)$.

Combining the last two theorems, we arrive at the following description of the angular semi-modules in the plane.

THEOREM 7.6.7. *There exists a pair of orthogonal unit vectors u, v such that the angular semi-module Σ consists of all vectors $\xi u + \eta v$ with $\xi > f(\eta)$, where*

(i) *η varies over one of the sets $E(\eta > 0)$, $E(\eta < 0)$, or $E(-\infty < \eta < \infty)$,*

(ii) *the function $f(\eta)$ has real numbers or $-\infty$ as values, is subadditive and upper semi-continuous and satisfies the condition* $\lim\inf_{\eta\to 0} f(\eta) = 0$ *or* $-\infty$.

Conversely, every set of vectors satisfying these conditions is an angular semi-module in E_2.

We say that a function $f(\eta)$ satisfying condition (ii) is an *admissible subadditive function.* Thus, to every angular semi-module Σ in the plane corresponds a "restricted product" $[\Pi, f(\eta)]$ where Π is an angular semi-module of real numbers and $f(\eta)$ is an admissible subadditive function defined on Π. Conversely, every restricted product defines an angular semi-module of the plane. In the preferred system of coordinates, the boundary of Σ is made up of the curve $\xi = f(\eta)$ together with horizontal line segments corresponding to the discontinuities of $f(\eta)$. If $\Pi = E(\eta > 0)$, the positive u-axis is also part of the boundary. Since the vector $b = u$ of Theorem 7.6.1 ordinarily is not uniquely determined, the same holds for the preferred coordinate system (u, v) so that the same angular semi-module may correspond to infinitely many restricted products. We list some of the geometric properties of angular semi-modules in the following

THEOREM 7.6.8. *An angular semi-module in the plane is a simply-connected point-set. It is either the whole plane or a subset of a half-plane. It is indefinitely reducible in the sense of Definition 7.5.1.*

PROOF. To show that Σ is arc-wise connected we exhibit for any two elements $\xi_1 u + \eta_1 v$ and $\xi_2 u + \eta_2 v$ of Σ a polygon which connects them and is contained in Σ. The construction is possible because the upper semi-continuous function $f(\eta)$ has a finite least upper bound in the interval $\eta_1 \leqq \eta \leqq \eta_2$. If μ exceeds this bound, the polygon is made up of the following three—possibly degenerate—arcs:

$$
\begin{array}{llll}
\text{(i)} & \xi = \xi_1 + \vartheta(\mu - \xi_1), & \eta = \eta_1, & \\
\text{(ii)} & \xi = \mu, & \eta = \eta_1 + \vartheta(\eta_2 - \eta_1), & 0 \leqq \vartheta \leqq 1, \\
\text{(iii)} & \xi = \mu + \vartheta(\xi_2 - \mu), & \eta = \eta_2. &
\end{array}
$$

To prove that Σ is simply-connected, let C be a simple closed curve consisting of points in Σ; we have to show that a point z inside C (in the sense of the Jordan Curve Theorem) is necessarily contained in Σ. Indeed, consider the ray $z - \rho u$, $\rho \geqq 0$. This ray will intersect C at a point $z_0 = z - \rho_0 u$ of Σ, and by Theorem 7.6.4 this implies that $z = z_0 + \rho_0 u$ is in Σ.

If $f(\eta) \equiv -\infty$ then Σ is a half-plane or the whole plane according as Π is a proper or an improper semi-module. We can dismiss these cases as trivial. Again, if Π is a proper semi-module, say $E(\eta > 0)$, then Σ is restricted to the upper half-plane. The only case remaining in doubt is that in which $\Pi = E(-\infty < \eta < \infty)$ and $\lim\inf f(\eta) = 0$. Since $f(\eta) \not\equiv +\infty$, Theorem 6.3.1

shows that $f(\eta) \neq -\infty$, so that $f(\eta)$ is finite. Theorem 6.6.1 then gives the existence of two finite quantities α and β with $\alpha \leqq \beta$ such that $f(\eta) \geqq \beta\eta$ when $\eta > 0$ and $f(\eta) \geqq \alpha\eta$ when $\eta < 0$, so that Σ is contained in a sector with vertex at the origin and opening $\leqq \pi$.

Finally, let $z \in \Sigma$. Since Σ is an open set, the closure of which contains the origin, we can find a z_0 such that z_0 and $z - z_0$ are in Σ. This asserts that z is the sum of two elements of Σ so that z is reducible and Σ consequently indefinitely reducible. This completes the proof.

The extension of these results from two dimensions to n is a fairly simple matter. Let Σ_n be a given angular semi-module in E_n and choose a Cartesian coordinate system $(u_1, u_2, \cdots, u_n)$ in which the positive u_1-axis belongs to the closure of Σ_n. This is possible by virtue of Theorem 7.6.1. We then project Σ_n on the orthogonal $(n - 1)$-dimensional space $E_{n-1} = (u_2, \cdots, u_n)$. The set of all vectors $y = (u_2, \cdots, u_n)$ such that there is a vector $x = (u_1, u_2, \cdots, u_n)$ in Σ_n is an angular semi-module in E_{n-1} as observed above. We denote the projection of Σ_n by Σ_{n-1}. We then define

$$f(y) = f(u_2, \cdots, u_n) = \liminf u_1 \text{ when } (u_1, u_2, \cdots, u_n) \in \Sigma_n .$$

The values of this function on vectors in E_{n-1} are real numbers or $-\infty$. That it is subadditive and upper semi-continuous is proved as above and the same type of argument also gives $\lim \inf_{y\to 0} f(y) = 0$ or $-\infty$. Thus every angular semi-module in E_n gives rise to a restricted product $[\Sigma_{n-1}, f(y)]$ and vice versa.

Thus, *in order to characterize the angular semi-modules in n-dimensional euclidean space we have to determine* (i) *the angular semi-modules in* $(n - 1)$*-dimensional space and* (ii) *the admissible subadditive functions on such a semi-module in* E_{n-1}. The necessary tools for carrying through this recursive process have been given in the preceding discussion and we shall not go into further detail here.

The methods developed above can also be used for a study of closed semi-modules in E_n containing the origin. Theorem 7.6.1 obviously applies, but when we project on E_{n-1} the projection, which is a semi-module and contains the origin, is not necessarily closed but is merely a set F_σ. It is therefore more natural to assume at the outset that the original semi-module is a set F_σ since this property is preserved under projection. We desist from further indications.

References. Bergström [1], Hille [7, §2.5], Hille and Zorn [1], and J. Nielsen [1].

3. Topological Semi-Modules.

7.7 Zorn's category theorem. The emphasis placed on the study of angular semi-modules in the preceding paragraph is justified by a previously unpublished result due to Max Zorn.

THEOREM 7.7.1. *Let $\mathfrak{X}$ be a topological additive group in the sense of Definition* 1.8.1. *Let S be a semi-module in $\mathfrak{X}$. If S is of the second category at the zero-element and if S satisfies the condition of Baire, then* Int (S) = Int $(\bar{S})$ *and* Int (S) *is dense in S.*

PROOF. The proof is based upon the properties of Kuratowski's operator D (cf. sections 1.2 and 1.8). We recall that $D(X)$ is the set of all points of $\mathfrak{X}$ where the set X is of the second category, that $D(X)$ is closed, contained in $\bar{X}$, and equals the closure of its own interior. For typographical reasons we shall denote the closure of X by X^* in this proof. The symbol $X + Y$ denotes the set $\{x + y\}$, $x \in X, y \in Y$, and $-Y = \{-y\}, y \in Y$. The point of departure is the inclusion

$$D(X) - Y \subset D(X - Y) \tag{7.7.1}$$

which is proved by observing that if X is of the second category at the point a then $X - Y$ is of the second category at every point $a - y, y \in Y$.

By assumption $\theta \in D(S)$ and consequently also $\theta \in D(-S)$. We now set $X = S$, $Y = -S$, obtaining $D(S) + S \subset D(S + S)$ and

$$S \subset D(S) + S \subset D(S + S) = D(S)$$

so that $S \subset D(S)$. Using the properties of $D(X)$ mentioned above, we see that $S^* \subset [D(S)]^* = D(S) \subset S^*$ so that $D(S) = S^*$.

To simplify the notation we write $-S = R$ and denote the complement of S by T. We shall prove that $D(T) = T^*$. We note first that $R + T \subset T$. Indeed, if $x \in S$, $y \in T$, then the assumption that $-x + y \in S$ implies that $y = x + (-x + y) \in S$, which is impossible. We have then

$$T \subset D(R) + T \subset D(R + T) \subset D(T) \subset T^*$$

and $T \subset D(T)$ implies $T^* \subset [D(T)]^* = D(T)$ so that $D(T) = T^*$ as asserted.

We now bring in the assumption that S satifies the condition of Baire. One formulation of this property is the assertion that $D(S) \cap D(T)$ is non-dense so that $S^* \cap T^*$ is non-dense. This says that Int $(S^*) \cap$ Int $(T^*) = \emptyset$. But if an open set does not meet a given set, then it cannot meet its closure. Hence

$$\text{Int }(S^*) \cap [\text{Int }(T^*)]^* = \emptyset.$$

But

$$[\text{Int }(T^*)]^* = [\text{Int } D(T)]^* = D(T) = T^* \tag{7.7.2}$$

so that Int $(S^*) \subset$ Int (S). The opposite inclusion is trivial and consequently Int (S^*) = Int (S) which was the first assertion.

In (7.7.2) we may replace T by S by a similar argument. This gives

$$[\text{Int }(S)]^* \cap S = S^* \cap S = S,$$

so that the interior of S is dense in S. This completes the proof.

This theorem shows that any non-pathological semi-module, having the origin as a limit point and located in a topological additive group, differs from an angular semi-module by a non-dense frontier set. The structure of the latter remains to be determined. It is also fair to remark that our discussion in the preceding paragraph does not throw much light on the structure of an angular semi-module in a topological additive group, since we have restricted ourselves essentially to euclidean space.

CHAPTER VIII

ADDITION THEOREMS IN A BANACH ALGEBRA

8.1. Introduction. We shall now take up the main theme of these Lectures: *the theory of one-parameter semi-groups of endomorphisms* and the various ramifications and applications of this theory. The point of departure is the following problem:

Let $\mathfrak{X}$ be a complex Banach space, $\mathfrak{E}(\mathfrak{X})$ the corresponding Banach algebra of endomorphisms of $\mathfrak{X}$. Further let Σ be a given angular semi-module of real or complex numbers. Determine all functions $T(\zeta)$ on Σ to $\mathfrak{E}(\mathfrak{X})$ such that for all ζ_1 and ζ_2 in Σ we have

$$T(\zeta_1)T(\zeta_2) = T(\zeta_1 + \zeta_2). \tag{8.1.1}$$

If $T(\zeta)$ is a solution of this problem, we refer to $\mathfrak{S} = \{T(\zeta)\}$ as a *one-parameter semi-group of endomorphisms with parameter manifold* Σ (see Definition 7.3.6). We may regard $\mathfrak{S}$ as a representation of the semi-module Σ, but this interpretation is of no use for the following.

The first case which must be settled is that in which Σ is the interval $(0, \infty)$. This case is kept in the foreground throughout; while the complex module will play a basic role in the present chapter, proper complex semi-modules will not be considered until Chapter XIII where analytic semi-groups are studied.

Equation (8.1.1) is formally *the addition theorem of the exponential function.* Actually the classical exponential function is a special instance of our theory for if we take $\mathfrak{X} = Z_1$, the complex plane, and define $T(\xi)$ as the similitude $w = f(\xi)z$ where

$$f(\xi_1 + \xi_2) = f(\xi_1)f(\xi_2), \qquad 0 < \xi_1, \xi_2 < \infty, \tag{8.1.2}$$

then $\mathfrak{S} = \{T(\xi)\}$ is a one-parameter semi-group of linear transformations. It is well known that if $f(\xi)$ is supposed to be measurable, then either $f(\xi) \equiv 0$ or there exists a complex number α such that $f(\xi) = e^{\alpha\xi}$. Moreover, this semi-group may be extended to an analytical group defined for all real and complex values of the parameter since $e^{\alpha\zeta}$ is an entire function of ζ and satisfies (8.1.2) for all values of ζ.

It is then natural to expect that if care is taken to exclude non-measurable solutions of (8.1.1) as well as projections, $T(\zeta)$ will have the form $\exp(\zeta A)$ where A is an operator on $\mathfrak{X}$ to itself and the definite interpretation of the exponential function is left open for the time being. In analogy with the classical case of continuous groups, we shall refer to A as *the infinitesimal generator of* $\mathfrak{S}$.

These expectations are fulfilled to some extent at least. If $\Sigma = (0, \infty)$ and

$T(\xi)$ is bounded and measurable in the uniform or in the strong sense in $(0, \infty)$, then $T(\xi)$ is also continuous in the same sense for $0 < \xi < \infty$, but $T(\xi)$ does not necessarily tend to a limit when $\xi \to 0$. If $\lim_{\xi \to 0} T(\xi)$ exists it has to be an idempotent, that is, a projection operator which, for the purposes of the present discussion, we may assume to be the identical transformation. *If* $\lim_{\xi \to 0} T(\xi) = I$, *we have two sharply differentiated cases according as the limit exists in the uniform or in the strong sense.* These two cases are referred to hereinafter as the *uniform* and the *strong* cases respectively. The uniform case shows close analogy with the classical situation: *an infinitesimal generator* A *exists,* $A \in \mathfrak{E}(\mathfrak{X})$, *and* $T(\xi) = \exp(\xi A)$ where the exponential function is interpreted as in Chapter V. Since $\exp(\zeta A)$ is well defined for all complex ζ and satisfies (8.1.1), we see that $\mathfrak{S}$ *may be embedded in the analytical group* $\mathfrak{G} = \{\exp(\zeta A)\}$.

Save for the existence of a unique infinitesimal generator A, none of this holds in the strong case. A is now an *unbounded* linear transformation whose domain is merely dense in $\mathfrak{X}$ and the symbol $\exp(\xi A)$ must be redefined. Several new interpretations will be given. The function $T(\xi)$ is strongly continuous but usually not differentiable, much less analytic; if $T(\xi)$ can be extended to the complex plane as an analytic function, the extension $T(\zeta)$ defines an analytic semi-group whose parameter set Σ is a proper complex semi-module, that is, a subset of a half-plane and never the whole plane. The case in which $T(\xi)$ does not tend to a limit when $\xi \to 0$ agrees in most respects with the strong case and may be handled with the same methods.

Both cases present themselves in the applications, but the strong case is by far the most interesting to the analyst. It offers more difficult problems and calls for more refined analysis, but the resulting theory is also richer and shows greater variety.

8.2. Orientation. The present chapter is devoted to the uniform case and related questions. In the uniform case the underlying space $\mathfrak{X}$ plays no role and it is only the Banach algebra $\mathfrak{E}(\mathfrak{X})$ that matters. We can omit all reference to $\mathfrak{X}$ and formulate the problem for an arbitrary complex Banach algebra $\mathfrak{B}$:

PROBLEM A. *Determine all measurable functions* $f(\xi)$ *on* $(0, \infty)$ *to* $\mathfrak{B}$ *such that for all* ξ_1 *and* ξ_2 *in* $(0, \infty)$

$$f(\xi_1 + \xi_2) = f(\xi_1)f(\xi_2). \tag{8.2.1}$$

This problem, however, is capable of further generalization in several different directions. We list three such extensions.

PROBLEM B. *Determine all functions* $F(x)$ *on a complex Banach space* $\mathfrak{X}$ *to a complex Banach algebra* $\mathfrak{B}$ *which are measurable on rays and satisfy*

$$F(x + y) = F(x)F(y) \tag{8.2.2}$$

for all x *and* y *in a given cone.*

These functional equations have the nature of addition theorems and the two remaining problems generalize this feature of the question. If $G(\alpha, \beta)$ is a given analytic function of two complex variables and u, v are elements of a Banach algebra, the symbol $G(u, v)$ is to be interpreted as in section 5.17, that is, as a principal extension.

Problem C. *Determine all measurable functions $f(\xi)$ on real numbers to a Banach algebra $\mathfrak{B}$ such that*

$$f(\xi_1 + \xi_2) = G[f(\xi_1), f(\xi_2)] \tag{8.2.3}$$

for all ξ_1, ξ_2 and $\xi_1 + \xi_2$ in some interval $(0, \omega)$.

Problem D. *Determine all functions $F(x)$ on a complex Banach space $\mathfrak{X}$ to a complex Banach algebra $\mathfrak{B}$ which are measurable on rays and satisfy*

$$F(x + y) = G[F(x), F(y)] \tag{8.2.4}$$

for all x, y and $x + y$ in some sector.

Here are four problems of increasing generality all of which will be partly solved in the present chapter. It turns out that measurability with respect to a positive scalar variable ξ implies continuity for $\xi > 0$ but not the existence of a limit when $\xi \to 0$. In the present chapter we separate and determine the solutions which are continuous at the origin, $\xi = 0$ or $x = \theta$; they are holomorphic functions of ξ and analytic functions of x respectively. Problems A and C are treated in some detail; Problems B and D, which can be reduced to A and C respectively, are discussed quite briefly.

Problem A has been in the literature in one form or another since 1935 when D. S. Nathan made an attack on equation (8.1.1). He considered a group (or group germ) of linear transformations $T(\xi)$. No mention is made of the uniform topology, only strong continuity is assumed explicitly, but his additional assumption that $\| T(\xi_1 + \xi_2) - I \| \leqq \theta < 1$ for ξ_1 and ξ_2 in some interval induces uniform continuity. The question was reopened from the point of view of Banach algebras by M. Nagumo and K. Yosida in 1936. An independent discussion has been given by N. Dunford (1942, unpublished).

No study seems to have been made of Problem B though Theorem 5.18.1 must have been known to anybody who gave the question a passing thought. Problems C and D have been discussed by Dunford and Hille (abstract 1944).

The chapter is divided into four paragraphs, one for each problem. References are listed below.

References. Dunford and Hille [1], Nagumo [1], Nathan [1], Yosida [1].

1. Problem A

8.3. Measurability implies continuity. In the following we discuss the measurable solutions of Problem A. All such solutions which are defined for $\xi > 0$ are also continuous, but they separate into two classes according as they are also continuous for $\xi = 0$ or not. Solutions of the first class are analytic and are actually entire functions of ξ. Solutions of the second class are much more varied and do not differ essentially from the solutions of the strong problem mentioned in section 8.1. The further analysis of this class will therefore be postponed to Chapter IX.

Theorem 8.3.1. *Let $\mathfrak{B}$ be a real or complex Banach algebra which need not have a unit element. Let $f(\xi)$ be an everywhere defined measurable function on the interval $(0, \infty)$ to $\mathfrak{B}$ such that for $0 < \xi_1, \xi_2 < \infty$*

$$f(\xi_1 + \xi_2) = f(\xi_1)f(\xi_2). \tag{8.3.1}$$

Then $f(\xi)$ is continuous for all positive values of ξ.

Remark. Measurability is taken in the strong sense. It may of course be replaced by the equivalent condition that $f(\xi)$ is weakly measurable and almost separately valued. It should be noted that if $\mathfrak{B} = \mathfrak{E}(\mathfrak{X})$, then $f(\xi)$ is assumed to be uniformly measurable in the sense of Definition 3.3.2 (1).

Proof. Since $f(\xi)$ is everywhere defined and strongly measurable, $\| f(\xi) \|$ is finite and measurable in the sense of Lebesgue. From (8.3.1) we get the basic inequality

$$\log \| f(\xi_1 + \xi_2) \| \leqq \log \| f(\xi_1) \| + \log \| f(\xi_2) \|, \tag{8.3.2}$$

so that $\log \| f(\xi) \|$ is a measurable subadditive function of ξ in $I_+ = (0, \infty)$ and $\log \| f(\xi) \| \not\equiv +\infty$. By Theorem 6.4.1, $\log \| f(\xi) \|$ is bounded above in any interval $(\epsilon, 1/\epsilon)$. Hence $\| f(\xi) \|$ is a bounded measurable function in any such interval. Choose three numbers α, β, ξ such that $0 < \alpha < \beta < \xi$. Then

$$\int_\alpha^\beta f(\xi - \eta) f(\eta)\, d\eta$$

exists as a (B)-integral since the integrand is a bounded measurable function of η. By (8.3.1) the value of the integral is simply $(\beta - \alpha)f(\xi)$. For small values of ϵ

$$(\beta - \alpha)[f(\xi + \epsilon) - f(\xi)] = \int_\alpha^\beta [f(\xi + \epsilon - \eta) - f(\xi - \eta)] f(\eta)\, d\eta,$$

whence

$$(\beta - \alpha) \| f(\xi + \epsilon) - f(\xi) \| \leqq M \int_{\xi-\beta}^{\xi-\alpha} \| f(\tau + \epsilon) - f(\tau) \|\, d\tau,$$

where $M = \sup \| f(\eta) \|$ for $\alpha \leqq \eta \leqq \beta$. By Theorem 3.6.3 the right member tends to zero with ϵ. It follows that $f(\xi)$ is continuous for every $\xi > 0$ and the theorem is proved.

The result of this theorem is essentially the best of its kind. Thus the assumption that $f(\xi)$ is continuous for $\xi > 0$ does not imply the existence of $\lim_{\xi \to 0} f(\xi)$ or that $f(\xi)$ is differentiable for $\xi > 0$ or satisfies a Lipschitz condition of prescribed order or, finally, that $f(\xi)$ is the boundary-value of a holomorphic function. All these plausible extensions are disproved by the following counter example.

We take $\mathfrak{B} = \mathfrak{E}\{L_2(-\pi, \pi)\}$, that is, the Banach algebra of linear bounded transformation on the space $L_2(-\pi, \pi)$ to itself with the customary metric. The transformations $T(\xi)$ which takes $x(t) \sim \sum x_n e^{nit}$ into

$$x_\xi(t) \sim x_0 + \sum{}' \, | n |^{-\xi} \exp [\operatorname{sgn} n \cdot e^{|n|} i\xi] \, x_n \, e^{nit} \tag{8.3.3}$$

is clearly an element of $\mathfrak{E}\{L_2(-\pi, \pi)\}$ which satisfies (8.3.1) for $\xi > 0$. Here $\| T(\xi) \| \equiv 1$ and

$$\| T(\xi + \delta) - T(\xi) \| = \sup \left\{ 2\pi \sum_{n=1}^{\infty} n^{-2\xi} \, |n^{-\delta} \exp [e^n i\delta] - 1 \, |^2 [|x_n|^2 + | x_{-n} |^2] \right\}^{\frac{1}{2}}$$

for $2\pi \sum_{-\infty}^{\infty} | x_n |^2 \leqq 1$. Consider now the function

$$\varphi(u; \xi, \delta) = u^{-\xi} \, | u^{-\delta} \exp (e^u i\delta) - 1 \, |$$

for $1 \leqq u$, $0 \leqq \xi$, $0 < \delta \leqq 0.1$. If $u \geqq \log (1/\delta)$ then

$$\varphi(u; \xi, \delta) < 2[\log (1/\delta)]^{-\xi}.$$

Further,

$$\varphi(\log(1/\delta); \xi, \delta) > \tfrac{1}{2} \, [\log(1/\delta)]^{-\xi}$$

and when $1 \leqq u < \log (1/\delta)$ we have $\varphi(u; \xi, \delta) < u^{-\xi} e^u \delta$. Here the right-hand member is concave upwards in the interval; it consequently does not exceed the larger of $e\delta$ and $[\log(1/\delta)]^{-\xi}$. These estimates show that

$$\tfrac{1}{2}[\log(1/\delta)]^{-\xi} \leqq \| T(\xi + \delta) - T(\xi) \| < 2 \, [\log(1/\delta)]^{-\xi}. \tag{8.3.4}$$

From this inequality one sees that $T(\xi)$ is continuous in the uniform topology for $\xi > 0$ and does not satisfy any Lipschitz condition. If $T(\xi)$ had a uniform limit when $\xi \to 0$, it would have to be the identity which is the limit of $T(\xi)$ in the strong topology. But (8.3.4) holds also when $\xi = 0$, $T(0) = I$, and shows that the uniform limit does not exist. The estimates also show that $T(\xi)$ cannot be differentiable as this would call for a Lipschitz condition of order one. We note finally that $T(\xi)$ is the sum of two orthogonal transformations $T_1(\xi)$ and $T_2(\xi)$ where $T_2(\xi)[x]$ is obtained by restricting n to negative values in (8.3.3). It is a simple matter to show that $T_1(\xi)$ and $T_2(\xi)$ are analytic functions of ξ in the upper and the lower half-planes of the complex plane respectively. Neither is holomorphic on the real axis as may be shown by proving estimates analogous

to (8.3.4) for $\|T_k(\xi \pm i\eta) - T_k(\xi)\|$. It follows that $T(\xi)$ cannot define the boundary values of an analytic function on any interval of the positive real axis.

This example may be varied so that different moduli of continuity result. It is also possible to make $\lim_{\xi\to 0} \| T(\xi) \| = +\infty$. Examples of the latter type are to be found in Chapter XVII (see Theorem 17.5.3).

At this point we have the choice of elaborating the theory of equation (8.3.1) without further assumptions on $f(\xi)$ or singling out solutions with special properties. The first alternative leads to a theory not essentially different from that presented in Chapters IX and XI below in the so-called strong case. We have merely to interpret $f(\xi)$ as a linear operator acting on a suitably chosen (B)-space $\mathfrak{X}$. This may be taken as $\mathfrak{B}$ itself, the operation being left-hand multiplication of x by $f(\xi)$. The measurability assumptions on $f(\xi)$ as an element of $\mathfrak{B}$ will then imply strong measurability of the operator $f(\xi)$ and the results of the "strong" theory apply. The problem of constructing solutions of (8.3.1) which are uniformly continuous for $\xi > 0$ (but not for $\xi = 0$) will be considered briefly in Chapter XII (see section 12.4).

8.4. The exponential solutions. The only case in which an essentially different theory results is that in which

$$\lim_{\xi\to 0} f(\xi) = j \tag{8.4.1}$$

exists. We can either assume (8.4.1) outright or else introduce the assumption indirectly in an equivalent form. Both alternatives will be used in the following.

THEOREM 8.4.1. *If $f(\xi)$ is defined for $\xi > 0$ and satisfies (8.3.1) and (8.4.1) then j is an idempotent of $\mathfrak{B}$ and*

$$f(\xi) = jf(\xi) = f(\xi)j.$$

Further $f(\xi)$ is continuous for $\xi \geqq 0$ if $f(0) = j$ by definition.

PROOF. That j is an idempotent follows from the relation

$$j = \lim_{\xi+\eta\to 0} f(\xi + \eta) = \lim_{\xi\to 0,\eta\to 0} f(\xi)f(\eta) = j^2.$$

Formula (8.4.1) implies that $f(\xi)$ is bounded for small $\xi > 0$ and from this it follows that to every ω, $\omega > 0$, there is a finite $M(\omega)$ such that $\| f(\xi) \| \leqq M(\omega)$ for $0 \leqq \xi \leqq \omega$. Put

$$\lim_{\eta\to 0} f(\xi + \eta) = jf(\xi) = f(\xi)j \equiv g(\xi).$$

Here the limit exists uniformly with respect to ξ in $[0, \omega]$. It is clear that $g(\xi)$ satisfies (8.3.1) for $\xi \geqq 0$ and that $g(\xi) = jg(\xi) = g(\xi)j$, $g(0) = j$. This implies that $g(\xi)$ is right continuous for $\xi \geqq 0$. It is, a fortiori, measurable and hence continuous by Theorem 8.3.1. We have then for $0 < \xi - \eta \leqq \xi$, $0 \leqq \eta \leqq \delta(\epsilon)$,

$$\begin{aligned} \| f(\xi + \eta) - f(\xi) \| &\leqq \| f(\xi + \eta) - g(\xi) \| + \| f(\xi) - g(\xi - \eta) \| \\ &\quad + \| g(\xi) - g(\xi - \eta) \| \leqq 3\epsilon \end{aligned}$$

so that $f(\xi)$ is continuous and $f(\xi) \equiv g(\xi)$. This completes the proof. If $j = \theta$ we see that $f(\xi) \equiv \theta$.

THEOREM 8.4.2. *Under the assumptions of the preceding theorem there exists an element a of $\mathfrak{B}$ such that $a = ja = aj$ and*

$$f(\xi) = j + \sum_{n=1}^{\infty} \frac{\xi^n}{n!} a^n. \tag{8.4.2}$$

The series is absolutely convergent for all real (complex) values of ξ and satisfies (8.3.1) *for all such values.*

PROOF. We know that $j^2 = j$ and $f(\xi) = jf(\xi) = f(\xi)j$ for all $\xi > 0$. Let us introduce the subalgebra $\mathfrak{B}_0 = j\mathfrak{B}j$ in which j plays the role of unit element. Conversely, $\mathfrak{B}_0$ contains every x such that $jx = xj = x$ and from this one concludes readily that $\mathfrak{B}_0$ is closed. We have obviously $f(\xi) \in \mathfrak{B}_0$ for all $\xi > 0$. Since $f(\xi) \to j$ when $\xi \to 0$, there is a $\delta > 0$ such that $\| f(\xi) - j \| < 1$ for $0 < \xi < \delta$. For such values of ξ we define

$$f(-\xi) = j + \sum_1^{\infty} [j - f(\xi)]^n,$$

so that $f(\xi)f(-\xi) = f(-\xi)f(\xi) = j = f(0)$. Thus $f(-\xi)$ is the inverse of $f(\xi)$ in $\mathfrak{B}_0$. It is clear that $f(-\xi)f(-\eta)$ is the inverse of $f(\xi + \eta)$ for $0 < \xi, \eta, \xi + \eta < \delta$, so that $f(-\xi - \eta) = f(-\xi)f(-\eta) = f(-\eta)f(-\xi)$. Using (8.3.1) we can then define $f(\xi)$ for all negative values of ξ and the extension is easily shown to be unique. The inverse of x in $\mathfrak{B}_0$ lies in $\mathfrak{B}_0$ and is a continuous function of x; hence the extension lies in $\mathfrak{B}_0$ and is a continuous function of ξ.

Since $\mathfrak{B}_0$ is closed and $f(\xi)$ is continuous, the integral $\int_\alpha^\beta f(\tau)\, d\tau$ exists for finite values of α, β and is an element of $\mathfrak{B}_0$. Further

$$\lim_{\beta \to \alpha} \frac{1}{\beta - \alpha} \int_\alpha^\beta f(\tau)\, d\tau = f(\alpha)$$

for all values of α. Since $f(\alpha)$ has an inverse in $\mathfrak{B}_0$, it follows that $\int_\alpha^\beta f(\tau)\, d\tau$ has an inverse for small values of $\beta - \alpha$. Now

$$\int_\alpha^\beta f(\xi + \tau)\, d\tau - \int_\alpha^\beta f(\tau)\, d\tau = [f(\xi) - j] \int_\alpha^\beta f(\tau)\, d\tau$$

which gives

$$\frac{1}{\xi} \int_\beta^{\beta+\xi} f(\tau)\, d\tau - \frac{1}{\xi} \int_\alpha^{\alpha+\xi} f(\tau)\, d\tau = \frac{1}{\xi} [f(\xi) - j] \int_\alpha^\beta f(\tau)\, d\tau.$$

Here we choose β so near to α that the integral on the right has an inverse in $\mathfrak{B}_0$ whence

$$\frac{1}{\xi} [f(\xi) - j] = \left\{ \frac{1}{\xi} \int_\beta^{\beta+\xi} f(\tau)\, d\tau - \frac{1}{\xi} \int_\alpha^{\alpha+\xi} f(\tau)\, d\tau \right\} \left\{ \int_\alpha^\beta f(\tau)\, d\tau \right\}^{-1}. \tag{8.4.3}$$

When $\xi \to 0$ the right side tends to a limit since the first factor tends to $f(\beta) - f(\alpha)$. It follows that

$$\lim_{\xi \to 0} \frac{1}{\xi}[f(\xi) - j] \equiv a \tag{8.4.4}$$

exists and is an element of $\mathfrak{B}_0$. Hence we have for all real α, β

$$f(\beta) - f(\alpha) = a \int_\alpha^\beta f(\tau)\, d\tau$$

and, in particular,

$$f(\xi) = j + a \int_0^\xi f(\tau)\, d\tau \tag{8.4.5}$$

for all real values of ξ. By iterated substitution one obtains

$$f(\xi) = j + \frac{\xi}{1!}a + \frac{\xi^2}{2!}a^2 + \cdots + \frac{\xi^n}{n!}a^n + \frac{1}{n!}\int_0^\xi (\xi - \tau)^n f(\tau)\, d\tau.$$

Letting $n \to \infty$ we obtain formula (8.4.2).

If $\mathfrak{B}$ is a complex Banach algebra, we see that the series defines a holomorphic function so that $f(\xi)$ is an entire function of ξ. A simple computation shows that $f(\xi)$ satisfies (8.3.1) for all complex values of ξ.

The main points in the preceding proof are due to N. Dunford (unpublished). Formula (8.4.3), which is the crucial point of the proof, may also be used to prove the result under apparently weaker assumptions. We shall give a couple of such theorems.

THEOREM 8.4.3. *Let $f(\xi)$ be a solution of* (8.3.1) *defined for $\xi > 0$ which is measurable and* (B)*-integrable in every finite interval* $(0, \omega)$. *If*

$$\lim_{\xi \to 0} \frac{1}{\xi}\int_0^\xi f(\tau)\, d\tau = j \tag{8.4.6}$$

exists, then j is an idempotent, the limit in (8.4.4) *exists, and $f(\xi)$ is given by formula* (8.4.2).

PROOF. By Theorem 8.3.1, $f(\xi)$ is continuous for $\xi > 0$. We have

$$f(\xi)\frac{1}{\eta}\int_0^\eta f(\tau)\, d\tau = \frac{1}{\eta}\int_0^\eta f(\tau + \xi)\, d\tau.$$

When $\eta \to 0$, the left side tends to $f(\xi)j$, the right to $f(\xi)$. Hence $f(\xi) = f(\xi)j = jf(\xi)$ for $\xi > 0$. Further

$$j\frac{1}{\xi}\int_0^\xi f(\tau)\, d\tau = \frac{1}{\xi}\int_0^\xi f(\tau)\, d\tau$$

and in the limit this gives $j^2 = j$.

We consider the closed subalgebra $\mathfrak{B}_0 = j\mathfrak{B}j$ again and note that $\int_0^\beta f(\tau)\, d\tau$ is an element of $\mathfrak{B}_0$ which has an inverse for sufficiently small values of β. From this we conclude that (8.4.3) holds for $\alpha = 0$ and small values of β, so that the proof can be continued as above.

Assumption (8.4.1) may be introduced in a still more indirect manner. The following theorem is an example of such a procedure; it may be regarded as a generalization of the theorem of D. S. Nathan mentioned in section 8.2.

THEOREM 8.4.4. *Let $\mathfrak{B}$ be a real or complex Banach algebra of which $\mathfrak{B}_0$ is a closed subalgebra and let $\mathfrak{B}_0$ have a unit element j. Let $f(\xi)$ be a measurable function on the interval* $(0, \infty)$ *to $\mathfrak{B}_0$ which satisfies* (8.3.1) *for* $0 < \xi_1, \xi_2 < \infty$ *and let there exist a ξ_0 such that $f(\xi_0)$ has an inverse in $\mathfrak{B}_0$. Then there exists an element a of $\mathfrak{B}_0$ such that $f(\xi)$ is given by formula* (8.4.2).

REMARK. The assumption that $f(\xi_0)$ has an inverse in $\mathfrak{B}_0$ may be replaced by $\|f(\xi_0) - j\| < 1$. By Theorem 5.2.1 this implies the existence of the inverse, however. The basic assumption of D. S. Nathan was of this character.

PROOF. By Theorem 8.3.1, $f(\xi)$ is continuous for $\xi > 0$. If $0 < \xi < \xi_0$ we have

$$f(\xi_0) = f(\xi)f(\xi_0 - \xi) = f(\xi_0 - \xi)f(\xi)$$

whence it follows that $f(\xi)$ also has an inverse in $\mathfrak{B}_0$. Since $f(n\xi_0) = [f(\xi_0)]^n$ has an inverse for each positive integer n, it follows that $f(\xi)$ has an inverse in $\mathfrak{B}_0$ for all $\xi > 0$.

From this point on, the proof proceeds as in the case of Theorem 8.4.2. We establish formula (8.4.3) for $0 < \alpha < \beta$; this does not require advance knowledge of the existence of $\lim_{\xi \to 0} f(\xi)$. From (8.4.3) we conclude that this limit actually exists and equals j and that the limit in (8.4.4) exists. The validity of (8.4.2) is then obvious.

In conclusion it should be observed that the real difficulties in discussing the functional equation (8.3.1) are associated with the interval $(0, \infty)$. If the function $f(\xi)$ is supposed to satisfy this equation for all real values of ξ, then the assumption that $f(\xi)$ is measurable in an interval $(-\delta, \delta)$ is enough to force $f(\xi)$ to be of the form given by (8.4.2). A solution defined on $(0, \infty)$ is only exceptionally of this form, however, and this is the reason why we have to impose fairly severe restrictions on $f(\xi)$ in order to single out these solutions.

2. PROBLEM B

8.5. Solutions on Banach spaces. We shall extend the results of the preceding paragraph to functions on a Banach space $\mathfrak{X}$ to a Banach algebra $\mathfrak{B}$. Here it is desired to solve the functional equation

$$(8.5.1) \qquad F(x + y) = F(x)F(y) = F(y)F(x)$$

imposing as few a priori conditions on $F(x)$ as possible. It is clearly desirable that the domain of definition $\mathfrak{D}$ of $F(x)$ should have the property of containing $x + y$ whenever x and y are contained; in other words $\mathfrak{D}$ should be a semi-module in $\mathfrak{X}$. But in a (B)-space we have also scalar multiplication and in order to apply the results previously obtained we have to bring the scalars into play. It is convenient to assume that $\alpha x \in \mathfrak{D}$ for all $\alpha > 0$ whenever $x \in \mathfrak{D}$. This leads to

DEFINITION 8.5.1. *A set $\mathfrak{K}$ of $\mathfrak{X}$ is called an open (a finitely open) positive cone if* (i) *$\mathfrak{K}$ is an open set (a finitely open set in the sense of Definition* 4.3.1) *and* (ii) $x, y \in \mathfrak{K}$, $\alpha > 0$ *implies* $x + y, \alpha x \in \mathfrak{K}$.

If $F(x)$ is defined on $\mathfrak{K}$ and satisfies (8.5.1) there, then for fixed $x \in \mathfrak{K}$ the function $F(\xi x)$ is defined for $\xi > 0$ and satisfies (8.3.1).

THEOREM 8.5.1. *Let $F(x)$ be defined on a finitely open positive cone $\mathfrak{K}$ and satisfy* (8.5.1) *for $x, y \in \mathfrak{K}$. Suppose that for every fixed $x \in \mathfrak{K}$, $F(\xi x)$ is a measurable function of ξ on $(0, \infty)$. If $\mathfrak{X}_{(n)}$ is any finite-dimensional linear subspace of $\mathfrak{X}$, then $F(x)$ is continuous on $\mathfrak{X}_{(n)} \cap \mathfrak{K}$. In particular, if $\mathfrak{K}$ is itself finite-dimensional, then $F(x)$ is continuous on $\mathfrak{K}$.*

PROOF. The following simple proof is due to C. E. Rickart; it replaces an elaborate argument of the author's. Suppose that $\mathfrak{X}_{(n)}$ is an n-dimensional linear subspace of $\mathfrak{X}$ and that $x_1, \cdots, x_n$ are n linearly independent vectors in $\mathfrak{X}_{(n)} \cap \mathfrak{K}$. Then the set $\mathfrak{K}_n$ consisting of all vectors of the form $x = \xi_1 x_1 + \cdots + \xi_n x_n$ with $\xi_1 > 0, \cdots, \xi_n > 0$ is in $\mathfrak{X}_{(n)} \cap \mathfrak{K}$. Let $x^{(0)} = \xi_1^{(0)} x_1 + \cdots + \xi_n^{(0)} x_n$ be a fixed point of $\mathfrak{K}_n$. It is desired to show that $F(x)$ is continuous in $\mathfrak{K}_n$ at x_0. By Theorem 8.3.1, for each i, $F(\xi_i x_i)$ is a continuous function of ξ_i for $\xi_i > 0$. Observe also that if $x^{(k)} = \xi_1^{(k)} x_1 + \cdots + \xi_n^{(k)} x_n$, then $\lim_{k\to\infty} x^{(k)} = x^{(0)}$ is equivalent to $\lim_{k\to\infty} \xi_i^{(k)} = \xi_i^{(0)}$, $i = 1, \cdots, n$. Moreover $\prod_{i=1}^n y_i$ is a continuous function of $(y_1, \cdots, y_n)$ for $y_i \in \mathfrak{B}$. Since

$$F(\xi_1 x_1 + \cdots + \xi_n x_n) = \prod_{i=1}^n F(\xi_i x_i),$$

it follows that the left member is a continuous function of $(\xi_1, \cdots, \xi_n)$ for $\xi_i > 0$. Therefore $F(x)$ is continuous in $\mathfrak{K}_n$ at x_0. Every point x in $\mathfrak{X}_{(n)} \cap \mathfrak{K}$ being interior to a suitably chosen set $\mathfrak{K}_n$, $F(x)$ is continuous everywhere in $\mathfrak{X}_{(n)} \cap \mathfrak{K}$ and the theorem is proved.

This theorem does not seem capable of much improvement. In particular, we cannot expect $F(x)$ to be continuous in $\mathfrak{K}$. Thus if $\mathfrak{B}$ has a unit element and $P(x)$ is a linear function on $\mathfrak{X}$ to $\mathfrak{B}$, then $\exp[P(x)]$ satisfies (8.5.1) for all x and $\exp[P(\xi x)] = \exp[\xi P(x)]$ is an entire function of ξ, but $\exp[P(x)]$ is not necessarily a continuous function of x since $P(x)$ does not have to be continuous. It is also clear that in general $\lim_{\xi\to 0} F(\xi x)$ need not exist for any $x \neq \theta$. We can

get somewhat further, however, by assuming the existence of the latter limit.

THEOREM 8.5.2. *Let $F(x)$ be defined and satisfy* (8.5.1) *for $x \in \mathfrak{K}$ where $\mathfrak{K}$ is now an open positive cone. If*

$$\lim_{\xi \to 0} F(\xi x) \tag{8.5.2}$$

exists for all x in $\mathfrak{K}$, then the limit is an idempotent j of $\mathfrak{B}$, independent of x, and there exists a function $P(x)$ such that $P(x) = jP(x) = P(x)j$ and

$$F(x) = j + \sum_{n=1}^{\infty} \frac{1}{n!}[P(x)]^n. \tag{8.5.3}$$

Here $P(x)$ may be defined for all x in $\mathfrak{X}$ as an additive, real-homogeneous function which is continuous if and only if the limit in (8.5.2) *exists uniformly with respect to x in some sphere.*

PROOF. By virtue of Theorem 8.4.2

$$F(\xi x) = j(x) + \sum_{n=1}^{\infty} \frac{\xi^n}{n!}[P(x)]^n, \qquad x \in \mathfrak{K},$$

where $[j(x)]^2 = j(x)$ and $P(x) = j(x)P(x) = P(x)j(x)$. We have first to show that $j(x)$ is independent of x. From

$$F(\alpha x)F(\beta y) = F(\beta y)F(\alpha x) = F(\alpha x + \beta y),\ \alpha, \beta > 0,$$

we get by letting $\alpha \to 0$

$$j(x)F(\beta y) = F(\beta y)j(x) = F(\beta y)$$

whence by letting $\beta \to 0$

$$j(x)j(y) = j(y)j(x) = j(y).$$

Reversing the order of the limiting processes we get $j(x)j(y) = j(x)$ so that $j(x) = j(y) = j$.

It remains to determine the properties of $P(x)$. Using the identity $F(\xi(x + y)) = F(\xi x)F(\xi y)$, substituting the power series in ξ, and equating coefficients of the first power of ξ, we get $P(x + y) = P(x) + P(y)$. We have also $P(\xi x) = \xi P(x)$ for $\xi > 0$. If $\mathfrak{K}$ is not the whole space, $P(x)$ may be extended from $\mathfrak{K}$ to $\mathfrak{X}$ as an additive, real-homogeneous function. Since $\mathfrak{K}$ is supposed to be open we can find a closed sphere $\| x - x_0 \| \leqq \rho$ which belongs to $\mathfrak{K}$. If $y \in \mathfrak{X}$ we can express y uniquely as $y = \alpha(x - x_0)$ where $\| x - x_0 \| = \rho$ and $\alpha > 0$. We then define $P(y) = \alpha[P(x) - P(x_0)]$. It may be shown that $P(\alpha y_1 + \beta y_2) = \alpha P(y_1) + \beta P(y_2)$ for all y_1, y_2 and real α, β, that the new definition of $P(y)$ agrees with the old one in $\mathfrak{K}$, and finally that the extension is independent of the choice of x_0 and ρ. It is clear that the series (8.5.3) satisfies the functional equation for all values of x and y.

Suppose now that $P(x)$ is continuous so that $\|P(x)\| \leqq M\|x\|$. We have then $\|F(\xi x) - j\| \leqq \exp[\xi M\|x\|] - 1$ for $\xi > 0$; consequently the limit in (8.5.2) exists uniformly with respect to x in any finite sphere. Suppose conversely that the limit exists uniformly with respect to x in the sphere $\mathfrak{S}: \|x - x_0\| < \rho$. We can then find a fixed β such that for every $x \in \mathfrak{S}$ we have $\|F(\tau x) - j\| \leqq \frac{1}{2}$, $0 < \tau \leqq \beta$, and consequently also

$$\left\| \beta^{-1} \int_0^\beta F(\tau x)\, d\tau - j \right\| \leqq \tfrac{1}{2}.$$

We can then apply formula (8.4.3) obtaining

$$P(x) = [F(\beta x) - j]\left\{\int_0^\beta F(\tau x)\, d\tau\right\}^{-1},$$

the norm of which does not exceed $(1/2\beta)[\|j\| + 1]$ in $\mathfrak{S}$. $P(x)$ being additive, real-homogeneous and bounded in a sphere is consequently bounded and hence continuous. This completes the proof of the theorem.

If $\mathfrak{B}$ is a real (B)-space, then $P(x)$, being additive and real-homogeneous, is actually linear, but this is no longer necessarily the case for complex (B)-spaces. In such a space there are always solutions of (8.5.1) defined and continuous for all x which are nowhere analytic in x. This occurs even in the simplest case $\mathfrak{X} = \mathfrak{B} = Z_1$ where $F(\zeta) = \exp[\alpha\xi + \beta\eta)]$, $\zeta = \xi + i\eta$, α and β arbitrary complex numbers, defines a solution of $F(\zeta_1)F(\zeta_2) = F(\zeta_1 + \zeta_2)$ which is clearly continuous but not analytic in ζ unless $\beta = \alpha i$.

3. Problem C

8.6. Addition theorems. The functional equations studied in the preceding paragraphs are addition formulas. Classical analysis presents us with a large number of such formulas and it is natural to ask if other addition theorems than that of the exponential function may be subjected to abstract analysis. This general question has been attacked by Nelson Dunford and the present writer; a brief account of the main results will be given in the remainder of this chapter.

In the present account we restrict ourselves to the case of a single function and an addition formula of the form

$$f(\xi_1 + \xi_2) = G[f(\xi_1), f(\xi_2)], \tag{8.6.1}$$

where $G(\alpha, \beta)$ is an analytic function of α and β such that $G(\alpha, \beta) = G(\beta, \alpha)$. In classical analysis $G(\alpha, \beta)$ is ordinarily supposed to be a rational or an algebraic function. We shall not make this assumption as it does not lead to any simplification of the, essentially local, problem which we are considering.

Classical function theory is concerned with the existence and properties of numerically-valued solutions of functional equations of the type represented by (8.6.1). The problem still has a meaning for vector-valued functions; we have merely to interpret the right-hand side of (8.6.1) properly and section 5.17 shows how this should be done.

Let $\mathfrak{B}$ be a complex Banach algebra with unit element e. Let $G(\alpha, \beta)$ be a symmetric analytic function of α and β which is holomorphic if both variables are in a certain domain Δ of the complex plane. Let u and v be two commuting elements of $\mathfrak{B}$ whose spectra are located in Δ; more precisely, u and v shall belong to the domain $\mathfrak{D}(\Delta)$ of Theorem 5.17.3. We then define

$$G(u, v) = \frac{1}{(2\pi i)^2} \int_{\Gamma_v} \int_{\Gamma_u} G(\alpha, \beta) R(\alpha; u) R(\beta; v)\, d\alpha\, d\beta, \tag{8.6.2}$$

where Γ_u and Γ_v are oriented envelopes of $\sigma(u)$ and $\sigma(v)$ in Δ.

Suppose now that $f(\xi)$ is a function on real numbers to $\mathfrak{B}$ such that for all ξ under consideration $f(\xi_1)f(\xi_2) = f(\xi_2)f(\xi_1)$ and $f(\xi) \in \mathfrak{D}(\Delta_0)$ where Δ_0 is an arbitrary domain such that $\bar{\Delta}_0 \subset \Delta$. Then by definition

$$G[f(\xi_1), f(\xi_2)] = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} G(\alpha, \beta) R[\alpha; f(\xi_1)] R[\beta; f(\xi_2)]\, d\alpha\, d\beta, \tag{8.6.3}$$

where Γ_1 and Γ_2 are fixed envelopes of $\bar{\Delta}_0$ in Δ.

We shall extend Theorem 8.3.1 to the solutions of Problem C. To this end it is convenient to assume that $G(u, v)$ is uniformly continuous on the range of $f(\xi)$. The most natural assumption is a Lipschitz condition of order one. Such a condition is automatically satisfied, for instance, if $G(\alpha, \beta)$ is an entire function and $f(\xi)$ is bounded, or if $f(\xi)$ is restricted to a closed set interior to the domain $\mathfrak{D}_\Delta$ of Theorem 5.17.1, or, finally, if the values of $f(\xi)$ belong to a compact subset of $\mathfrak{D}(\Delta)$.

THEOREM 8.6.1. *Let $f(\xi)$ be a measurable function of ξ on the open interval $(0, \omega)$ to $\mathfrak{B}$ satisfying* (8.6.1) *for all ξ_1 and ξ_2 in the interval and such that $f(\xi_1)$ and $f(\xi_2)$ commute. Let $\mathfrak{R}_\epsilon$ denote the range of $f(\xi)$ for $0 < \epsilon < \xi < \omega$ where $\mathfrak{R}_\epsilon$ is bounded and $\mathfrak{R}_\epsilon \subset \mathfrak{D}(\Delta_\epsilon)$, $\bar{\Delta}_\epsilon \subset \Delta$. Suppose that for every ϵ, $0 < \epsilon < \omega$, there is a finite positive M_ϵ such that for u_1, u_2, v in $\mathfrak{R}_\epsilon$*

$$\| G(u_1, v) - G(u_2, v) \| \leqq M_\epsilon \| u_1 - u_2 \|. \tag{8.6.4}$$

Then $f(\xi)$ is continuous in $(0, \omega)$.

PROOF. Let $0 < \alpha \leqq \eta \leqq \beta < \xi < \omega$ and note that

$$f(\xi) = G[f(\xi - \eta), f(\eta)],$$

whence

$$(\beta - \alpha) f(\xi) = \int_\alpha^\beta G[f(\xi - \eta), f(\eta)]\, d\eta$$

and

$$(\beta - \alpha) \| f(\xi + \epsilon) - f(\xi) \| \leqq \int_\alpha^\beta \| G[f(\xi + \epsilon - \eta), f(\eta)] - G[f(\xi - \eta), f(\eta)] \| \, d\eta.$$

If $\delta = \min (\alpha, \xi - \beta, \xi + \epsilon - \beta)$, then the right member is dominated by

$$M_\delta \int_\alpha^\beta \| f(\xi + \epsilon - \eta) - f(\xi - \eta) \| \, d\eta = M_\delta \int_{\xi-\beta}^{\xi-\alpha} \| f(\tau + \epsilon) - f(\tau) \| \, d\tau,$$

which tends to zero with ϵ since $f(\xi)$ is bounded and measurable. Hence $f(\xi)$ is actually continuous in $(0, \omega)$.

In passing let us note that the requirement that $f(\xi_1)$ and $f(\xi_2)$ shall commute is frequently implied by the functional equation and need not be assumed explicitly.

The counter example of section 8.3 shows that we cannot expect $f(\xi)$ to approach a limit when $\xi \to 0$ or to have any stronger properties of continuity in $(0, \omega)$. The situation is entirely different if $f(\xi)$ satisfies (8.6.1) in an interval containing $\xi = 0$ or is supposed to tend to a finite limit when $\xi \to 0$. We shall consider the latter case in some detail. If $\lim_{\epsilon\to 0} f(\epsilon)$ exists and equals an element a of $\mathfrak{D}(\Delta)$, then

$$\lim_{\epsilon \to 0} f(\xi + \epsilon) = \lim_{\epsilon \to 0} G[f(\xi), f(\epsilon)] = G[f(\xi), a]$$

exists for $0 < \xi < \omega$. From the fact that $f(\xi)$ has a right-hand limit everywhere, it follows that $f(\xi)$ is right-hand continuous (even continuous) except possibly in a countable set. Since $f(\xi)$ in particular is measurable, it has to be continuous everywhere in $(0, \omega)$.

8.7. Holomorphic solutions. Just as in the case of Problem A, continuity at the origin implies analyticity. We set

$$G_1(\alpha, \beta) = \frac{\partial}{\partial \alpha} G(\alpha, \beta),$$

$$Q(\alpha, \beta, \gamma) = \frac{G(\alpha, \gamma) - G(\beta, \gamma)}{\alpha - \beta}, \qquad \alpha, \beta, \gamma \in \Delta.$$

The latter is evidently a holomorphic function of α, β, γ in Δ. We can then define $Q(u, v, w)$ for u, v, w in $\mathfrak{D}(\Delta)$ by the obvious triple resolvent integral, assuming u, v, w to commute, and

$$Q(u, v, w)(u - v) = G(u, w) - G(v, w). \tag{8.7.1}$$

LEMMA 8.7.1. *If $f(\xi) \in \mathfrak{D}(\Delta)$, if $f(\xi)$ is continuous for $0 \leqq \xi \leqq \omega$ and $0 \leqq \eta, \zeta \leqq \omega$, then uniformly with respect to ζ*

$$\lim_{\xi \to \eta} Q[f(\xi), f(\eta), f(\zeta)] = G_1[f(\eta), f(\zeta)].$$

Proof. First note that since $f(\xi)$ is continuous in $[0, \omega]$, the range of $f(\xi)$ is a closed connected compact set $\mathfrak{R} \subset \mathfrak{D}(\Delta)$. If $\Phi = \bigcup \sigma(x)$, $x \in \mathfrak{R}$, then Φ is a closed subset of Δ and if Γ is an oriented envelope of Φ in Δ, then $R[\alpha; f(\xi)]$ is a continuous function of (α, ξ) for $\alpha \in \Gamma$, $0 \leqq \xi \leqq \omega$. There is consequently a finite positive $M = M(\Gamma)$ such that

$$\| R[\alpha; f(\xi)] \| \leqq M(\Gamma), \qquad \alpha \in \Gamma, 0 \leqq \xi \leqq \omega. \tag{8.7.2}$$

Further

$$\lim_{\xi \to \eta} R[\alpha; f(\xi)] = R[\alpha; f(\eta)]$$

uniformly with respect to α on Γ. It follows that uniformly in ζ, $0 \leqq \zeta \leqq \omega$

$$\begin{aligned}
&\lim_{\xi \to \eta} Q[f(\xi), f(\eta), f(\zeta)] \\
&= -\frac{1}{8\pi^3 i} \int_\Gamma \int_\Gamma \int_\Gamma Q(\alpha, \beta, \gamma) R[\alpha; f(\eta)] R[\beta; f(\eta)] R[\gamma; f(\zeta)] \, d\alpha \, d\beta \, d\gamma \\
&= \frac{1}{8\pi^3 i} \int_\Gamma \int_\Gamma \int_\Gamma Q(\alpha, \beta, \gamma) \frac{R[\alpha; f(\eta)] - R[\beta; f(\eta)]}{\alpha - \beta} R[\gamma; f(\zeta)] \, d\alpha \, d\beta \, d\gamma \\
&= \frac{1}{8\pi^3 i} \int_{\Gamma_1} \int_{\Gamma_1} d\beta \, d\gamma R[\gamma; f(\zeta)] \left\{ \int_\Gamma \frac{Q(\alpha, \beta, \gamma) R[\alpha; f(\eta)]}{\alpha - \beta} \, d\alpha \right. \\
&\qquad\qquad \left. - R[\beta; f(\eta)] \int_\Gamma \frac{Q(\alpha, \beta, \gamma)}{\alpha - \beta} \, d\alpha \right\}.
\end{aligned}$$

Here Γ and Γ_1 are oriented envelopes of Φ in Δ and Γ_1 is interior to Γ. Now

$$\int_\Gamma \frac{Q(\alpha, \beta, \gamma)}{\alpha - \beta} \, d\alpha = 2\pi i Q(\beta, \beta, \gamma) = 2\pi i G_1(\beta, \gamma),$$

so

$$\begin{aligned}
\lim_{\xi \to \eta} Q[f(\xi), f(\eta), f(\zeta)] &= -\frac{1}{4\pi^2} \int_{\Gamma_1} \int_{\Gamma_1} G_1(\beta, \gamma) R[\beta; f(\eta)] R[\gamma; f(\zeta)] \, d\beta \, d\gamma + U \\
&= G_1[f(\eta), f(\zeta)] + U.
\end{aligned}$$

Here

$$\begin{aligned}
U &= \frac{1}{8\pi^3 i} \int_{\Gamma_1} \int_{\Gamma_1} d\beta \, d\gamma R[\gamma; f(\zeta)] \int_\Gamma \frac{Q(\alpha, \beta, \gamma)}{\alpha - \beta} R[\alpha; f(\eta)] \, d\alpha \\
&= \frac{1}{8\pi^3 i} \int_{\Gamma_1} d\gamma R[\gamma; f(\zeta)] \int_\Gamma d\alpha R[\alpha; f(\eta)] \int_{\Gamma_1} \frac{Q(\alpha, \beta, \gamma)}{\alpha - \beta} \, d\beta.
\end{aligned}$$

But for $\alpha \in \Gamma$ the last integral is zero so $U = \theta$. This completes the proof of the lemma.

We shall now prove that continuity of a solution at $\xi = 0$ implies the existence of derivatives of all orders. We give the proof under the added assumption that $G_1[f(0), f(0)]$ has an inverse. This assumption has the effect of cutting out

solutions involving other idempotents than e. It would be sufficient, however, to assume the existence of the inverse in the subalgebra determined by $f(\xi)$. It is tacitly assumed in the following that this algebra is commutative.

THEOREM 8.7.1. *If $f(\xi)$, having values in $\mathfrak{D}(\Delta)$, is a continuous solution of (8.6.1) for $0 \leqq \xi \leqq \omega$ and if $G_1[f(0), f(0)]$ has an inverse in $\mathfrak{B}$, then $f(\xi)$ has derivatives of all orders and*

$$f'(\xi) = G_1[f(0), f(\xi)]f'(0). \tag{8.7.3}$$

If $g(\xi)$ is any solution of (8.6.1) which is continuous in $[0, \omega]$ and commutes with $f(\xi)$ and if $g(0) = f(0)$, $g'(0) = f'(0)$, then $g(\xi) \equiv f(\xi)$.

PROOF. From the contour integral definition of $G_1[f(0), f(\zeta)]$ we see that it is continuous in ζ and hence

$$\lim_{\alpha\to 0} \frac{1}{\alpha}\int_0^\alpha G_1[f(0), f(\zeta)]\,d\zeta = G_1[f(0), f(0)].$$

Thus we may fix $\alpha < \omega$ so the integral on the left has an inverse in $\mathfrak{B}$. From Lemma 8.7.1 we have

$$\lim_{\xi\to 0} \frac{1}{\alpha}\int_0^\alpha Q[f(\xi), f(0), f(\zeta)]\,d\zeta = \frac{1}{\alpha}\int_0^\alpha G_1[f(0), f(\zeta)]\,d\zeta.$$

Hence

$$\lim_{\xi\to 0}\left\{\frac{1}{\alpha}\int_0^\alpha Q[f(\xi), f(0), f(\zeta)]\,d\zeta\right\}^{-1} = \left\{\frac{1}{\alpha}\int_0^\alpha G_1[f(0), f(\zeta)]\,d\zeta\right\}^{-1}. \tag{8.7.4}$$

By formula (8.7.1) we have

$$[f(\xi) - f(\eta)]\,Q[f(\xi), f(\eta), f(\zeta)] = G[f(\xi), f(\zeta)] - G[f(\eta), f(\zeta)], \tag{8.7.5}$$

whence

$$\begin{aligned}
\frac{1}{\xi}[f(\xi) - f(0)]\frac{1}{\alpha}\int_0^\alpha & Q[f(\xi), f(0), f(\zeta)]\,d\zeta \\
&= \frac{1}{\alpha\xi}\int_0^\alpha \{G[f(\xi), f(\zeta)] - G[f(0), f(\zeta)]\}\,d\zeta \\
&= \frac{1}{\alpha\xi}\int_0^\alpha [f(\xi+\zeta) - f(\zeta)]\,d\zeta \\
&= \frac{1}{\alpha\xi}\left\{\int_\alpha^{\alpha+\xi} f(\tau)\,d\tau - \int_0^\xi f(\tau)\,d\tau\right\} \to \frac{1}{\alpha}[f(\alpha) - f(0)]
\end{aligned} \tag{8.7.6}$$

as $\xi \to 0$. Thus (8.7.4) and (8.7.6) give the existence of

$$\lim_{\xi\to 0}\frac{1}{\xi}[f(\xi) - f(0)] = \left\{\frac{1}{\alpha}\int_0^\alpha G_1[f(0), f(\zeta)]\,d\zeta\right\}^{-1}\frac{1}{\alpha}[f(\alpha) - f(0)]$$

so $f(\xi)$ is differentiable at $\xi = 0$. Applying the lemma once more we have

$$\frac{1}{\eta}[f(\xi+\eta)-f(\xi)] = \frac{1}{\eta}\{G[f(\eta), f(\xi)] - G[f(0), f(\xi)]\}$$
$$= \frac{1}{\eta}[f(\eta)-f(0)]\,Q[f(\eta), f(0), f(\xi)] \to f'(0)\,G_1[f(0), f(\xi)]$$

uniformly for $0 \leqq \xi \leqq \omega$. This establishes formula (8.7.3). That the two factors in the derivative commute follows from the fact that the factors on the left in formula (8.7.5) commute.

Once the existence of the first derivative has been established, that of the higher derivatives follows by easy steps. It is enough to indicate the argument for the second derivative. We have

(8.7.7) $$f'(\xi) = f'(0)\frac{1}{(2\pi i)^2}\int_\Gamma\int_\Gamma G_1(\alpha, \beta)R[\alpha; f(0)]R[\beta; f(\xi)]\,d\alpha\,d\beta.$$

Now when $\eta \to 0$

$$\frac{1}{\eta}\{R[\beta; f(\xi+\eta)] - R[\beta; f(\xi)]\} \to -f'(\xi)\{R[\beta; f(\xi)]\}^2,$$

where we have used formula (5.5.2) and the fact that $f'(\xi)$ commutes with $R[\gamma; f(\zeta)]$ which is obvious from (8.7.7). Here the limit exists uniformly with respect to β on Γ. Hence

$$f''(\xi) = -f'(0)f'(\xi)\frac{1}{(2\pi i)^2}\int_\Gamma\int_\Gamma G_1(\alpha, \beta)R[\alpha; f(0)]\{R[\beta; f(\xi)]\}^2\,d\alpha\,d\beta$$

and a contour integral argument of familiar type shows that this expression equals

$$f'(0)f'(\xi)G_{1,1}[f(0), f(\xi)],$$

where

$$G_{1,1}(\alpha, \beta) = \frac{\partial^2}{\partial\alpha\,\partial\beta}G(\alpha, \beta)$$

and $G_{1,1}(u, v)$ is defined by the usual contour integral for $u, v \in \mathfrak{D}(\Delta)$. We see in particular that

$$f''(0) = [f'(0)]^2 G_{1,1}[f(0), f(0)]$$

and hence is uniquely determined by $f(0)$ and $f'(0)$. Similarly it is shown that all higher derivatives exist and are uniquely determined by $f(0)$ and $f'(0)$.

There is consequently a formal power series in ξ associated with $f(\xi)$ determined by the initial values $f(0)$ and $f'(0)$. We shall not attempt to prove the convergence of this series by direct estimates. In the most important case considered below the convergence follows from other considerations.

8.8. Uniqueness of the solution. The uniqueness assertion of Theorem 8.7.1 still remains to be proved. Here we shall use the Lipschitz condition of Theorem 8.6.1 which is easily verified in the present situation.

Suppose that $g(\xi)$ is continuous in $[0, \omega]$, has values in $\mathfrak{D}(\Delta)$ and commutes with $f(\eta)$. From the integral representation of the function $Q[f(\xi), g(\xi), f(\eta)]$ combined with (8.7.2), we see that there is a finite $K = K(f, g)$ such that for $0 \leqq \xi, \eta \leqq \omega$

$$\| Q[f(\xi), g(\xi), f(\eta)] \| \leqq K, \qquad \| Q[g(\eta), f(\eta), g(\xi)] \| \leqq K.$$

Since

$$G[f(\xi), f(\eta)] - G[g(\xi), f(\eta)] = Q[f(\xi), g(\xi), f(\eta)][f(\xi) - g(\xi)],$$

we have

$$\| G[f(\xi), f(\eta)] - G[g(\xi), f(\eta)] \| \leqq K \| f(\xi) - g(\xi) \| \tag{8.8.1}$$

and a similar inequality in which f, g and ξ, η are interchanged.

We suppose now that $f(\xi)$ and $g(\xi)$ are continuous solutions of (8.6.1) and that $f(0) = g(0), f'(0) = g'(0)$. We know then that they have derivatives of all orders and that $f^{(n)}(0) = g^{(n)}(0)$ for all n. Placing $h(\xi) = f(\xi) - g(\xi)$ we have

$$\begin{aligned} h(\xi + \eta) &= f(\xi + \eta) - g(\xi + \eta) \\ &= G[f(\xi), f(\eta)] - G[g(\xi), g(\eta)] \\ &= G[f(\xi), f(\eta)] - G[g(\xi), f(\eta)] + G[g(\xi), f(\eta)] - G[g(\xi), g(\eta)], \end{aligned}$$

so from (8.8.1)

$$\| h(\xi + \eta) \| \leqq K\{\| f(\xi) - g(\xi) \| + \| f(\eta) - g(\eta) \|\} = K\{\| h(\xi) \| + \| h(\eta) \|\}$$

whence

$$\| h(2\xi) \| \leqq 2K \| h(\xi) \|.$$

By repeated use of this inequality we get

$$\| h(\xi) \| \leqq (2K)^m \| h(\xi 2^{-m}) \|, \qquad m = 1, 2, 3, \cdots. \tag{8.8.2}$$

Now consider

$$g_n(\xi) = g(\xi) - \sum_{\nu=0}^{n} g^{(\nu)}(0) \frac{\xi^\nu}{\nu!}$$

and let

$$M_n[g] = \max \| g_n^{(n)}(\xi) \| = \max \| g^{(n)}(\xi) - g^{(n)}(0) \|$$

in the interval $[0, \omega]$. Since

$$g_n(\xi) = \int_0^\xi \cdots \int_0^{\xi_{n-2}} \int_0^{\xi_{n-1}} g_n^{(n)}(\xi_n)\, d\xi_n\, d\xi_{n-1} \cdots d\xi_1$$

we have

$$\| g_n(\xi) \| \leqq \frac{\xi^n}{n!} M_n[g], \qquad 0 \leqq \xi \leqq \omega.$$

If $M_n[g]$ is replaced by $M_n[f]$, the same inequality is satisfied by

$$f_n(\xi) = f(\xi) - \sum_{\nu=0}^{n} f^{(\nu)}(0) \frac{\xi^\nu}{\nu!}.$$

In the following M_n denotes the larger of the two quantities $M_n[f]$ and $M_n[g]$. It should be noted that the polynomial in ξ is the same in $f_n(\xi)$ as in $g_n(\xi)$.

From

$$\| h(\xi) \| = \| f(\xi) - g(\xi) \| = \| f_n(\xi) - g_n(\xi) \| \leqq \| f_n(\xi) \| + \| g_n(\xi) \|,$$

we see that

$$\| h(\xi) \| \leqq 2 \frac{\xi^n}{n!} M_n. \tag{8.8.3}$$

Combination of this estimate with (8.8.2) gives

$$\| h(\xi) \| \leqq \frac{2}{n!} M_n \, \omega^n (2^{1-n} K)^m, \qquad 0 \leqq \xi \leqq \omega, \; m, n = 1, 2, 3, \cdots. \tag{8.8.4}$$

Here we fix n so large that $2^{1-n}K < 1$. Since m is independent of n and may be taken arbitrarily large, we see that $h(\xi) \equiv 0$ and so $g(\xi) \equiv f(\xi)$, $0 \leqq \xi \leqq \omega$. This completes the proof of Theorem 8.7.1.

Suppose now that $\varphi(\zeta)$ is an analytic scalar function such that (i) $\varphi(\zeta)$ is not a constant, (ii) $\varphi(\zeta) = \sum_0^\infty \alpha_n \zeta^n$ convergent for $|\zeta| < \rho$, (iii) $\varphi(\zeta) \in \Delta$, and (iv)

$$\varphi(\zeta_1 + \zeta_2) = G[\varphi(\zeta_1), \varphi(\zeta_2)] \tag{8.8.5}$$

for $|\zeta_1|$, $|\zeta_2|$, $|\zeta_1 + \zeta_2| < \rho$. Differentiating $\varphi(\zeta) = G[\varphi(\zeta), \varphi(0)]$ we get $\varphi'(\zeta) = \varphi'(\zeta) G_1[\varphi(\zeta), \varphi(0)]$; by (i) this implies that

$$G_1[\varphi(\zeta), \varphi(0)] \equiv 1$$

and, in particular,

$$G_1[\varphi(0), \varphi(0)] = 1. \tag{8.8.6}$$

Differentiating (8.8.5) by parts with respect to ζ_1 and putting $\zeta_1 = 0$ in the result, one gets [cf. formula (8.7.3)]

$$\varphi'(\zeta) = \varphi'(0) G_1[\varphi(0), \varphi(\zeta)]$$

whence $\varphi'(0) \neq 0$. But if $\varphi(\zeta)$ satisfies the conditions stated above, so does $\varphi(\alpha\zeta)$ for any value of α provided ρ be replaced by $\rho/|\alpha|$. We can consequently normalize $\varphi(\zeta)$ by assuming (v) that $\varphi'(0) = 1$.

Suppose now that $f(\xi)$ is a continuous, and hence differentiable, solution of (8.6.1) such that $f(0) = \varphi(0)e$, $f'(0) = a$. Then

$$G_1[f(0), f(0)] = G_1[\varphi(0), \varphi(0)]e = e$$

and the existence of an inverse is trivial. On the other hand, the function $\varphi(a\zeta)$ is defined by the series

$$\varphi(a\zeta) = \sum_0^\infty \alpha_n a^n \zeta^n$$

at least for $|\zeta| < \rho/\|a\|$ and for such values of ζ we have also $\sigma[\varphi(a\zeta)] = \varphi[\zeta\sigma(a)] \in \Delta$ since $|\sigma(a)| \leqq \|a\|$. From the construction of the series and the properties mentioned, it follows that it satisfies (8.6.1) for $|\zeta_1|$, $|\zeta_2|$, $|\zeta_1 + \zeta_2| < \rho/\|a\|$. Further $\varphi(0) = \varphi(0)e$, $\varphi'(0) = \alpha_1 a = a$. As a corollary of Theorem 8.7.1 we then obtain

THEOREM 8.8.1. *Let $\varphi(\zeta)$ satisfy conditions* (i)$-$(v) *listed above and let $f(\xi)$ be a continuous, and hence differentiable, solution of* (8.6.1) *such that $f(0) = \varphi(0)e, f'(0) = a$. Then*

$$f(\xi) \equiv \varphi(a\xi)$$

in the common interval of definition of the two functions. Thus $f(\xi)$ admits of an extension to the complex ζ-plane, satisfying (8.6.1), *and this extension, if defined by $f(\zeta) = \varphi(a\zeta)$, is holomorphic in some neighborhood of $\zeta = 0$.*

We sum up the main results of our study of Problem C in

THEOREM 8.8.2. *If the addition formula*

$$f(\zeta_1 + \zeta_2) = G[f(\zeta_1), f(\zeta_2)]$$

is known to have an analytic scalar solution $\varphi(\zeta)$, holomorphic in some neighborhood of $\zeta = 0$, if $\varphi(\zeta)$ is not a constant and has values in a domain Δ where $G(\alpha, \beta)$ is holomorphic, then the formula has solutions having values in an arbitrary prescribed complex Banach algebra with unit element e. If $f(\xi)$ is such a solution defined and continuous for $0 \leqq \xi \leqq \omega$, if $f(\xi) \in \mathfrak{D}(\Delta)$ and $f(0) = \varphi(0)e$, then $f(\xi)$ is not merely differentiable but actually analytic and in some neighborhood of $\zeta = 0$ we have $f(\zeta) = \varphi(a\zeta)$ where $a = f'(0)$.

For the validity of our argument and even of the conclusions it is quite essential that $\varphi(0)$ belongs to the domain of holomorphism of $G(\alpha, \beta)$. This excludes from consideration some important addition theorems, for instance, those of $\cos \zeta$ and $\cos \operatorname{am} \zeta$, where $\varphi(0)$ belongs to a singular manifold of $G(\alpha, \beta)$. In these cases it is preferable to replace the given addition theorem by a pair of addition theorems for the function and its first derivative. The discussion of such and more general systems, we defer to another occasion.

4. PROBLEM D

8.9. Solutions in G-power series. We shall restrict ourselves to a fairly simple special case of Problem D which may be reduced to Problem C plus the theory of G-power series.

Let $\mathfrak{X}$ be a complex Banach space, $\mathfrak{B}$ a complex Banach algebra with unit element e, and let $G(\alpha, \beta)$ be a symmetric analytic function which is holomorphic if α and β are in a domain Δ of the complex plane. We suppose that the functional equation (8.8.5) has a scalar solution $\varphi(\zeta)$ satisfying conditions (i) to (v) of section 8.8. We take $\rho = 1$.

THEOREM 8.9.1. *Let $F(x)$ be a function on $\mathfrak{X}$ to $\mathfrak{B}$ defined for $\| x \| < 1$ and having values in $\mathfrak{D}(\Delta)$. It is supposed that $F(\xi x)$ is a continuous function of ξ for $0 \leqq \xi < 1$ if x is fixed, $\| x \| < 1$. Further*

$$\lim_{\xi \to 0} F(\xi x) = \varphi(0)e \tag{8.9.1}$$

for every fixed x and

$$F(x + y) = G[F(x), F(y)] \tag{8.9.2}$$

if $\| x \|, \| y \|, \| x + y \| < 1$. Then there exists a linear function $P(x)$ on $\mathfrak{X}$ to $\mathfrak{B}$ such that

$$F(x) = \varphi[P(x)] \equiv \sum_{0}^{\infty} \alpha_n [P(x)]^n \tag{8.9.3}$$

for $\| x \| < 1$. The series is an F- or a G-power series according as $P(x)$ is bounded or not. In the former case $F(x)$ is analytic in $\| x \| < 1$. A necessary and sufficient condition that $P(x)$ be bounded is that the limit in (8.9.1) *exists uniformly with respect to x in $\| x \| < \delta$.*

PROOF. For fixed x the function $F(\xi x) = f(\xi)$ satisfies the conditions of Theorem 8.8.1. There is consequently an $a = P(x)$ such that $F(\zeta x) = \varphi[\zeta P(x)]$ at least for $| \zeta | < 1/ \| P(x) \|$. It remains merely to discuss the properties of $P(x)$. We shall prove that $F(x)$ is G-differentiable and

$$\delta F(x; h) = P(h)\, G_1[F(\theta), F(x)]. \tag{8.9.4}$$

Since $P(h)$ is the derivative of $\varphi[\zeta P(h)]$ with respect to ζ at $\zeta = 0$ and we have $G_1[F(\theta), F(\theta)] = e$, the formula is true for $x = \theta$. From Lemma 8.7.1 we get

$$\begin{aligned} \frac{1}{\zeta} [F(x + \zeta h) - F(x)] &= \frac{1}{\zeta} \{G[F(\zeta h), F(x)] - G[F(\theta), F(x)]\} \\ &= \frac{1}{\zeta} [F(\zeta h) - F(\theta)]\, Q[F(\zeta h), F(\theta), F(x)] \\ &\to P(h)\, G_1[F(\theta), F(x)] \end{aligned}$$

when $\zeta \to 0$. This proves that $F(x)$ is G-differentiable for $\| x \| < 1$.

By Theorem 4.3.4, $\delta F(x; h)$ and in particular $\delta F(\theta; h) = P(h)$ are linear. From $F(\zeta x) = \varphi[\zeta P(x)]$ we get, differentiating with respect to ζ and placing $\zeta = 0$ in the result,

$$\delta^n F(\theta; h) = n!\,\alpha_n[P(h)]^n$$

which is clearly a homogeneous polynomial in h of degree n. It follows that (8.9.3) is the MacLaurin series of $F(x)$ which by Theorem 4.3.6 converges in the c-star about θ in the domain of G-differentiability. This implies that the series converges and represents $F(x)$ in the unit sphere. In order that the series be an F-power series it is evidently necessary and sufficient that $P(x)$ be bounded and hence continuous. If $P(x)$ is bounded, then (8.9.1) holds uniformly for $\|x\| \leqq 1 - \delta$. Conversely if the limit exists uniformly with respect to x in $\|x\| < \delta$, then $F(x)$ is continuous at $x = \theta$. From

$$F(x + h) = G[F(x), F(h)]$$

and the continuity of $G(u, v)$ with respect to u and v in $\mathfrak{D}(\Delta)$, it follows that $F(x)$ is continuous everywhere in $\|x\| < 1$. $F(x)$ is then F-differentiable and hence analytic in the unit sphere. This completes the proof of the theorem.

CHAPTER IX

SEMI-GROUPS IN THE STRONG TOPOLOGY

9.1. Orientation. The present chapter is devoted to the "strong case". We consider two problems corresponding to Problems A and B of the preceding chapter, appending a few remarks on Problem C.

The first question is the case of a *one-parameter semi-group* $\mathfrak{S} = \{T(\xi)\}$, $\xi > 0$, of linear bounded transformations on a complex (B)-space $\mathfrak{X}$ to itself so that

$$T(\xi_1 + \xi_2)x = T(\xi_1)[T(\xi_2)x]$$

for all $\xi_1, \xi_2 > 0$ and all $x \in \mathfrak{X}$.

The second question is that of a semi-group $\mathfrak{S} = \{T(x)\}$, $x \in \mathfrak{K}$, of linear bounded transformations on a complex (B)-space $\mathfrak{Y}$ to itself, the parameter manifold being an open positive cone $\mathfrak{K}$ in a (B)-space $\mathfrak{X}$ so that

$$T(x_1 + x_2)y = T(x_1)[T(x_2)y]$$

for all $x_1, x_2 \in \mathfrak{K}$ and all $y \in \mathfrak{Y}$. The second problem contains as a particular case what is conventionally known as an *n-parameter semi-group* in which case $\mathfrak{X}$ is a real euclidean n-dimensional space.

In the first problem, strong measurability of $T(\xi)$ for $\xi > 0$ together with boundedness of $\| T(\xi) \|$ on every closed interval interior to $(0, \infty)$ forces $T(\xi)$ to be strongly continuous for $\xi > 0$. If $\mathfrak{X}$ is a separable space, the boundedness condition is superfluous and strong measurability may be replaced by weak.

We introduce the *infinitesimal generator* A of $\mathfrak{S}$ defined as the strong limit of

$$A_\eta = \frac{1}{\eta}[T(\eta) - I]$$

whenever it exists. A is ordinarily an unbounded operator but its domain is dense in the union of the range spaces of $T(\alpha)$, $\alpha > 0$. If the latter is dense in $\mathfrak{X}$, then $T(\eta)x \to x$ when $\eta \to 0$ for all x. The main result is the "*exponential formula*"

$$(\mathrm{E}_1) \qquad T(\xi)x = \lim_{\eta \to 0} \exp(\xi A_\eta)\, x$$

for all x and all $\xi > 0$, which replaces the representation $T(\xi) = \exp(\xi A)$ of the "uniform case". Several other exponential formulas will be proved in Chapter XI which contains a detailed discussion of the properties of A, its resolvent $R(\lambda; A)$ and the relations between $T(\xi)$ and $R(\lambda; A)$ through the Laplace transformation. In connection with (E_1) see also E. L. Post [1] where $A = d/dt$.

In the second problem, strong measurability and boundedness on all "rays" of $\mathfrak{R}$ induces strong continuity of $T(x)$ on rays which may be extended to strong continuity on finite-dimensional subspaces. In this case we have a family of generators $\{A(x)\}$, $x \in \mathfrak{R}$, which form a semi-module $\mathfrak{A}$ with positive multipliers. $\mathfrak{A}$ corresponds to the Lie ring in the classical theory of continuous groups. In particular, we are able to determine the structure of all strongly measurable n-parameter semi-groups. They turn out to be direct products of commuting continuous one-parameter semi-groups.

The literature on one-parameter transformation groups is quite extensive, going back to M. H. Stone (1930) and J. von Neumann for the case of unitary groups in Hilbert space, the general case being treated by I. Gelfand (1939) and M. Fukamiya (1940). The earliest investigation having a direct bearing on the corresponding problem for semi-groups is a paper by B. de Sz. Nagy (1936), though some special semi-groups had been discussed by the present writer a few months earlier. Nagy's representation of groups of self-adjoint linear bounded transformations on a Hilbert space contains implicitly the results for semi-groups rediscovered by the author in 1938. Semi-groups analytic in a half-plane were investigated by the author in 1938 but the general case was not attacked until 1942. Extensions to the n-parameter case were announced in 1944. N. Dunford's contributions to semi-group theory started in 1938 when he proved the first general continuity theorem for one-parameter semi-groups (*strong measurability plus boundedness on* $(0, \omega)$ *implies strong right-hand continuity*) which he extended later to the n-parameter case. Several of Dunford's unpublished results are inserted below.

Groups and semi-groups of linear transformations $\{L(u)\}$, *multiplicative* in the parameter u, have been studied by N. P. Romanoff (1942). He works with function spaces F, normally without a topology, and presupposes that the transform $L(u)[f]$ is differentiable with respect to u for all $f \in F$. The transformation λ defined by

$$\lambda f = \left\{\frac{\partial}{\partial u} L(u)[f]\right\}_{u=1}$$

plays the role of infinitesimal generator. In one of his axiomatics $\lambda f \in F$ for all f which would seem to correspond to our "uniform case" though comparisons are not easily made due to the difference in approach. Romanoff has discovered a large number of general methods of constructing such multiplicative systems and his contributions to the operational calculus of the infinitesimal generator are very important.

Groups and semi-groups of linear transformations have also been studied from the point of view of partially ordered sets. In this connection we note the work of G. Birkhoff (1939) and L. Alaoglu on ergodic theorems in general semi-groups. More recently (1943) B. Vulich has investigated linear multiplicative operations on partially ordered spaces satisfying the axioms of Kantorovich and their analytic representation in some important special cases. These investigations appear to be rather remote from our line of approach so we shall not go beyond the mere mentioning of their existence.

There are two paragraphs: *One-Parameter Semi-Groups* and *Extensions*.

References. Alaoglu and Birkhoff [1], G. Birkhoff [4], Dunford [4, 5, 6, 9],

Dunford and Segal [1], Fukamiya [1], Gelfand [3], Hille [**4, 5, 6, 7, 9, 10**], v. Neumann [4], Post [1], Romanoff [**1, 2**], Stone [1], de Sz. Nagy [**1, 2**], and Vulich [**1, 2**].

1. One-parameter Semi-groups

9.2. Measurability and continuity. Let $\mathfrak{X}$ be a complex (B)-space, $\mathfrak{E}(\mathfrak{X})$ the complex (B)-algebra of bounded linear transformations on $\mathfrak{X}$ to itself, and let $T(\xi)$ be a function on positive numbers to $\mathfrak{E}(\mathfrak{X})$ such that

$$(9.2.1) \qquad T(\xi_1 + \xi_2)x = T(\xi_1)[T(\xi_2)x], \qquad 0 < \xi_1, \xi_2 < \infty, x \in \mathfrak{X}.$$

Thus $\mathfrak{S} = \{T(\xi)\}$ is a semi-group of operators in $\mathfrak{E}(\mathfrak{X})$.

It is to be expected that, in some sense or other, measurability of $T(\xi)$ implies continuity for $\xi > 0$. The proof given below presupposes that $\| T(\xi) \|$ is bounded for $0 < \epsilon \leqq \xi \leqq 1/\epsilon$. The boundedness of the norm is implied by weak measurability of $T(\xi)$ provided $\mathfrak{X}$ is a separable space. In the non-separable case, however, boundedness must be postulated as well as strong measurability instead of weak. This distinction between the two cases is possibly due to defects of the method.

Theorem 9.2.1. *$T(\xi)$ is strongly continuous for $\xi > 0$ if either $\mathfrak{X}$ is separable and $T(\xi)$ is weakly measurable or $\| T(\xi) \|$ is bounded in each interval $[\epsilon, 1/\epsilon]$ and $T(\xi)$ is strongly measurable.*

Proof. We start by proving that the first set of assumptions implies the second one. If $\mathfrak{X}$ is a separable space, part (1) of Theorem 3.3.1 shows that $T(\xi)$ is strongly measurable whenever it is weakly measurable. It follows that $\| T(\xi)x \|$ is measurable Lebesgue for each x. But if $\{x_n\}$ is dense on the unit sphere in $\mathfrak{X}$ then $\| T(\xi) \| = \sup \| T(\xi)x_n \|$ is measurable as the supremum of a countable set of measurable functions. $\| T(\xi_1 + \xi_2) \| \leqq \| T(\xi_1) \| \| T(\xi_2) \|$ shows that $\log \| T(\xi) \|$ is a subadditive measurable function, never equal to $+\infty$; it follows from Theorem 6.4.1 that $\log \| T(\xi) \|$ is bounded above in every interval $[\epsilon, 1/\epsilon]$. Thus the second set of assumptions are satisfied.

We can then proceed as in the proof of Theorem 8.3.1. We choose four numbers α, β, ξ, η such that $0 < \alpha < \eta < \beta < \xi$ and an ϵ so small that also $\beta < \xi - \epsilon$. We have

$$T(\xi)x = T(\eta)[T(\xi - \eta)x];$$

the right side being independent of η is certainly integrable with respect to η so that

$$(\beta - \alpha)[T(\xi \pm \epsilon) - T(\xi)]x = \int_\alpha^\beta T(\eta)\{[T(\xi \pm \epsilon - \eta) - T(\xi - \eta)]x\}\, d\eta.$$

If $\| T(\tau) \| \leqq M$ when $\alpha \leqq \tau \leqq \beta$, the norm of the integrand does not exceed $M \| [T(\xi \pm \epsilon - \eta) - T(\xi - \eta)]x \|$ which is a bounded measurable function of η. Hence

$$(\beta - \alpha) \| [T(\xi \pm \epsilon) - T(\xi)]x \| \leqq M \int_{\xi-\beta}^{\xi-\alpha} \|[T(\tau \pm \epsilon) - T(\tau)]x \| \, d\tau.$$

By Theorem 3.6.3 the right member tends to zero with ϵ. It follows that $T(\xi)x$ is continuous for $\xi > 0$ and the theorem is proved.

The strong continuity of $T(\xi)$ implies that $\| T(\xi)x \|$ is continuous for fixed x. From this one infers without difficulty that $\| T(\xi) \|$ is lower semi-continuous and a fortiori measurable. Whether or not $\| T(\xi) \|$ is necessarily continuous is an open question.

Each of the two hypotheses in the second set of conditions in Theorem 9.2.1 is evidently necessary for the desired conclusion, but we may well question their degree of independence also in the case of a non-separable space. It is conceivable that weak measurability suffices in all cases. The following theorem due to N. Dunford shows that it would be sufficient to prove weak continuity, but throws no further light on the moot question.

THEOREM 9.2.2. *Weak continuity of $T(\xi)$ on $(0, \infty)$ implies strong continuity.*

PROOF. If $T(\xi)$ is weakly continuous for $\alpha \leqq \xi \leqq \beta$, then to every $x \in \mathfrak{X}$, $x^* \in \mathfrak{X}^*$ there is a finite $M(\alpha, \beta, x, x^*)$ such that $| x^*[T(\xi)x] | \leqq M(\alpha, \beta, x, x^*)$ when $\alpha \leqq \xi \leqq \beta$. By the principle of uniform boundedness (apply Theorems 2.12.3 and 2.12.2 in this order) this implies first that $\| T(\xi)x \| \leqq M(\alpha, \beta, x)$ and secondly that $\| T(\xi) \| \leqq M(\alpha, \beta)$ when $\alpha \leqq \xi \leqq \beta$. Thus $\| T(\xi) \|$ is bounded in any closed interval interior to $(0, \infty)$. Next we note that $T(\xi)$ is weakly measurable. Less on the surface lies the fact that $T(\xi)x$ is separably-valued. This is proved as follows. Let $\{\xi_n\}$ be the positive rational numbers and consider, for a fixed x, the set $\mathfrak{M}_x$ the elements of which are of the form $\sum (\alpha_n + i\beta_n)T(\xi_n)x$ where the α's and β's are arbitrary rational numbers, zero for large values of n. Let $\mathfrak{R}_x$ be the set $\{T(\xi)x\}$, $0 < \xi < \infty$, and suppose that there is an element $T(\xi_0)x$ of $\mathfrak{R}_x$ which is not a limit point of $\mathfrak{M}_x$. We can then find a linear bounded functional x_0^* such that $x_0^*[T(\xi_n)x] = 0$ for all n but $x_0^*[T(\xi_0)x] = 1$. On the other hand, if $\xi_{n_k} \to \xi_0$, then by the weak continuity of $T(\xi)$ we have $x_0^*[T(\xi_{n_k})x] \to x_0^*[T(\xi_0)x]$. This contradiction shows that $\mathfrak{R}_x$ belongs to the closure of the countable set $\mathfrak{M}_x$ and this implies that $\mathfrak{R}_x$ is separable so that $T(\xi)x$ is a separably-valued function of ξ. But a separably-valued, weakly measurable function is strongly measurable. Since in addition $\| T(\xi) \|$ is bounded on interior intervals, $T(\xi)$ is strongly continuous as asserted.

That strong continuity of $T(\xi)$ does not imply uniform continuity is shown by the counter example in section 16.2.

9.3. The first exponential formula. Once the strong continuity of $T(\xi)$ has been established, we may proceed to a further study of the structure of

$T(\xi)$. This requires the introduction of the *infinitesimal generator* A of the semi-group $\mathfrak{S} = \{T(\xi)\}$. We define

$$A_\eta = \frac{1}{\eta}[T(\eta) - I], \qquad \eta > 0, \tag{9.3.1}$$

$$Ax = \lim_{\eta \to 0} A_\eta x \tag{9.3.2}$$

whenever the limit exists. The set of elements x for which Ax exists will be denoted by $\mathfrak{D}(A)$. It is clearly a linear subspace of $\mathfrak{X}$ and we start by showing that $\mathfrak{D}(A)$ never reduces to the zero element.

Suppose now that $T(\xi)$ is strongly continuous for $\xi > 0$ and that x is of the form

$$x = x_{\alpha,\beta} = \int_\alpha^\beta T(\tau)y\, d\tau, \qquad y \in \mathfrak{X},\ 0 < \alpha < \beta < \infty. \tag{9.3.3}$$

Then

$$\begin{aligned} A_\eta x_{\alpha,\beta} &= \frac{1}{\eta}\int_\alpha^\beta [T(\eta) - I]T(\tau)y\, d\tau \\ &= \frac{1}{\eta}\int_\alpha^\beta [T(\tau + \eta) - T(\tau)]y\, d\tau \\ &= \frac{1}{\eta}\int_\beta^{\beta+\eta} T(\sigma)y\, d\sigma - \frac{1}{\eta}\int_\alpha^{\alpha+\eta} T(\sigma)y\, d\sigma \\ &\to [T(\beta) - T(\alpha)]y \end{aligned}$$

when $\eta \to 0$. Hence every element of this form belongs to $\mathfrak{D}(A)$. This observation is due to N. Dunford (unpublished).

Let $\mathfrak{X}_\alpha = T(\alpha)[\mathfrak{X}]$ be the range of the transformation $T(\alpha)$, $\alpha > 0$. We have clearly

$$\mathfrak{X}_\alpha \supset \mathfrak{X}_\beta \qquad \text{if } \alpha < \beta \tag{9.3.4}$$

and define

$$\mathfrak{X}_0 = \textstyle\bigcup_\alpha \mathfrak{X}_\alpha, \qquad \alpha > 0. \tag{9.3.5}$$

Thus $\mathfrak{X}_0$ *is the least linear hull of the range-spaces of* $\mathfrak{S}$.

THEOREM 9.3.1. *If $T(\xi)$ is strongly continuous for $\xi > 0$, then $\mathfrak{D}(A)$ is dense in $\mathfrak{X}_0$, the two sets have the same closure, and the range of A also belongs to $\bar{\mathfrak{X}}_0$.*

PROOF. If $x \in \mathfrak{X}_0$ there exists an $\alpha > 0$ and a $y \in \mathfrak{X}$ such that $x = T(\alpha)y$. If $x_{\alpha,\beta}$ is defined by formula (9.3.3) then $\lim_{\beta\to\alpha} [1/(\beta - \alpha)]x_{\alpha,\beta} = x$, that is, every point of $\mathfrak{X}_0$ belongs to the closure of $\mathfrak{D}(A)$. Conversely, if $x \in \mathfrak{D}(A)$,

then $\lim_{\eta \to 0} T(\eta)x = x$ so that $x \in \bar{\mathfrak{X}}_0$. It follows that the closures of $\mathfrak{X}_0$ and $\mathfrak{D}(A)$ are identical. Finally, if $x \in \mathfrak{D}(A)$, then $A_\eta x \in \bar{\mathfrak{X}}_0$ and so does Ax.

THEOREM 9.3.2. *If $x \in \mathfrak{D}(A)$, then*

$$\frac{d}{d\xi} T(\xi)x = AT(\xi)x = T(\xi)Ax.$$

PROOF. We have

$$\frac{1}{\eta}[T(\xi + \eta) - T(\xi)]x = A_\eta T(\xi)x = T(\xi)A_\eta x,$$

where the last member tends to $T(\xi)Ax$ when $\eta \to 0$ whence the theorem follows.

We now introduce the operator

$$\exp[(\xi - \alpha)A_\eta]\, T(\alpha) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (\xi - \alpha)^n A_\eta^n\, T(\alpha) \tag{9.3.6}$$

which is an element of $\mathfrak{E}(\mathfrak{X})$ since A_η is bounded. Here α is fixed, $\alpha > 0$, ξ is arbitrary, $\xi \geqq \alpha$, η is positive, and the main question before us is what happens to the operator when $\eta \to 0$. We prove first that the operator stays bounded. This is proved by the following

THEOREM 9.3.3. *Let $\| T(\xi) \| \leqq M$ for $0 < \alpha \leqq \xi \leqq \max(\alpha + 1, 2\alpha)$; then for all $\xi \geqq \tau > \alpha$, $\eta > 0$, we have*

$$\| \exp[(\xi - \tau)A_\eta]\, T(\tau) \| \leqq \begin{cases} M^{\tau/\alpha} \exp\left\{\dfrac{\xi - \tau}{\eta}[M^{\eta/\alpha} - 1]\right\}, & \alpha \geqq 1, \\[2ex] M^{1+\tau-\alpha} \exp\left\{\dfrac{\xi - \tau}{\eta}[M^{\eta} - 1]\right\}, & \alpha < 1, \end{cases} \tag{9.3.7}$$

provided $M \geqq 1$. For $M < 1$ we divide each estimate by M.

PROOF. From the given estimate it follows that $\| T(\xi) \|$ for $\xi \geqq \alpha$ is dominated by

$$M^{\xi/\alpha}, \quad M^{\xi/\alpha - 1}, \quad M^{1+\xi-\alpha}, \quad \text{and} \quad M^{\xi-\alpha}$$

according as $\alpha \geqq 1$, $M \geqq 1$; $\alpha \geqq 1$, $M < 1$; $\alpha < 1$, $M \geqq 1$; and $\alpha < 1$, $M < 1$. We then use the identity

$$\begin{aligned} \exp[(\xi - \tau)A_\eta]\, T(\tau) &= e^{-(\xi-\tau)/\eta} \exp\left\{\frac{\xi - \tau}{\eta} T(\eta)\right\} T(\tau) \\ &\equiv e^{-(\xi-\tau)/\eta} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\xi - \tau}{\eta}\right)^n T(\tau + n\eta) \end{aligned} \tag{9.3.8}$$

of which the desired estimates are an immediate consequence. It follows in particular that for $0 < \eta \leqq 1$, $M \geqq 1$, $\xi \geqq \tau \geqq \alpha$,

$$\| \exp[(\xi - \tau)A_\eta]\, T(\tau) \| \leqq M^{1+\tau} \exp[(\xi - \tau)(M - 1)]. \tag{9.3.9}$$

For $M < 1$ we may replace the right member by 1. These estimates show that the operator stays bounded when $\eta \to 0$. That it has a strong limit will be shown in the next theorem (the *first exponential formula*) which is the main result of this chapter. For the proof we need two lemmas, the use of which was proposed by G. Pólya. The proof of the first is left to the reader.

LEMMA. 9.3.1. $\sum_{n=0}^{\infty} (n - w)^2(n!)^{-1}w^n = we^w$.

LEMMA 9.3.2. *If $w > 0$ and the summation is extended only over those values of n for which $|n - w| > N$, then*

$$e^{-w} \sum' \frac{w^n}{n!} < N^{-2}w.$$

PROOF. We have

$$we^w > \sum' (n - w)^2 \frac{w^n}{n!} > N^2 \sum' \frac{w^n}{n!}.$$

THEOREM 9.3.4. *If $T(\xi)$ is strongly continuous for $\xi > 0$, then for every $x \in \mathfrak{X}$ and every $\xi \geqq \alpha > 0$*

$$\lim_{\eta \to 0} \| \exp [(\xi - \alpha)A_\eta] T(\alpha)x - T(\xi)x \| = 0. \tag{9.3.10}$$

If $T(\xi)$ is uniformly continuous for $\xi > 0$

$$\lim_{\eta \to 0} \| \exp [(\xi - \alpha)A_\eta] T(\alpha) - T(\xi) \| = 0. \tag{9.3.11}$$

In both cases the limit exists uniformly with respect to ξ in every finite interval $[\alpha, \beta]$.

PROOF. Let α, β, ξ, x be given, $0 < \alpha \leqq \xi \leqq \beta < \infty$. We suppose that $\| T(\xi) \| \leqq M$ for $\alpha \leqq \xi \leqq \max (\alpha + 1, 2\alpha)$. Let $\mu(\delta, x)$ be the rectified modulus of continuity of $T(\xi)x$ in $[\alpha, \beta]$ so that $\| T(\xi_1)x - T(\xi_2)x \| \leqq \mu(\delta, x)$ whenever $\alpha \leqq \xi_1, \xi_2 \leqq \beta$, $|\xi_1 - \xi_2| \leqq \delta$. From (9.3.8) one gets

$$\exp [(\xi - \alpha)A_\eta] T(\alpha)x - T(\xi)x = e^{-(\xi-\alpha)/\eta} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\xi - \alpha}{\eta}\right)^n [T(\alpha + n\eta) - T(\xi)]x,$$

whence

$$\| \exp [(\xi - \alpha)A_\eta] T(\alpha)x - T(\xi)x \|$$
$$\leqq e^{-(\xi-\alpha)/\eta} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\xi - \alpha}{\eta}\right)^n \| [T(\alpha + n\eta) - T(\xi)] x \| .$$

Let S_1 denote the sum of the terms of the latter series in which $|\alpha + n\eta - \xi| \leqq \delta$ and S_2 the rest. Here $S_1 < \mu(\delta, x)$ and

$$S_2 \leqq e^{-(\xi-\alpha)/\eta} \sum' \frac{1}{n!}\left(\frac{\xi - \alpha}{\eta}\right)^n \{ \| T(\xi)x \| + \| T(\alpha + n\eta)x \| \} \equiv S_{21} + S_{22}.$$

Now S_{21} is the product of $\| T(\xi)x \|$ with an exponential series to which Lemma 9.3.2 applies with $w = (\xi - \alpha)/\eta$, $N = \delta/\eta$ so that

$$S_{21} < \eta(\xi - \alpha)\delta^{-2} \| T(\xi)x \|.$$

The discussion of S_{22} is slightly more involved. Restricting ourselves to the typical case $M \geqq 1$ we have $\| T(\alpha + n\eta)x \| \leqq M^{1+n\eta} \| x \|$, regardless of the value of α, whence

$$S_{22} \leqq M \| x \| e^{-(\xi-\alpha)/\eta} \sum{}' \frac{1}{n!} \left(\frac{\xi - \alpha}{\eta} M^{\eta} \right)^n.$$

Here we have to take $w = [(\xi - \alpha)/\eta]M^{\eta}$, but the range of summation is not symmetric with respect to w and, unless η is quite small, it may contain integers from a neighborhood of w. For this reason we choose an $\eta_0 \leqq 1$ so small that

$$(\beta - \alpha)(M^{\eta_0} - 1) = \tfrac{1}{2}\delta.$$

On this assumption the set of integers n for which

$$\left| n - \frac{\xi - \alpha}{\eta} M^{\eta} \right| > \frac{\delta}{2\eta}, \qquad 0 < \eta \leqq \eta_0,$$

contains the set occurring in S_{22}. By Lemma 9.3.2 we have then

$$\begin{aligned} S_{22} &< \exp \left\{ \frac{\xi - \alpha}{\eta} (M^{\eta} - 1) \right\} \frac{\xi - \alpha}{\eta} M^{\eta} \frac{4\eta^2}{\delta^2} M \| x \| \\ &< 4\eta(\beta - \alpha)\delta^{-2} M^2 e^{(\beta-\alpha)(M-1)} \| x \|. \end{aligned}$$

Hence for $\alpha \leqq \xi \leqq \beta$, $1 \leqq M$, $0 < \eta \leqq \eta_0$,

$$\begin{aligned} \| \exp [(\xi - \alpha)A_{\eta}]\, T(\alpha)x - T(\xi)x \| &< \mu(\delta, x) \\ &\quad + \eta(\beta - \alpha)\delta^{-2}[M^{\beta+1} + 4M^2 e^{(\beta-\alpha)(M-1)}] \| x \|. \end{aligned} \tag{9.3.12}$$

If $M < 1$, the second term on the right should be divided by M. From this estimate in which δ is arbitrarily small, we conclude that (9.3.10) holds uniformly in $[\alpha, \beta]$. If $T(\xi)$ is uniformly continuous for $\xi > 0$ and not merely strongly continuous, we may suppress x everywhere in (9.3.12) replacing $\mu(\delta, x)$ by $\mu(\delta)$, the modulus of continuity of $T(\xi)$ in $[\alpha, \beta]$, and from this we see that (9.3.11) holds uniformly with respect to ξ in $[\alpha, \beta]$.

The proof of the preceding theorem is modeled upon an argument devised by G. Szegö for the special case of the semi-group of right-hand translations in $C[0, \infty]$. It replaces a more complicated and less powerful argument based on the resolvents of A and A_{η} previously found by the author. For the important case in which the least closed linear hull of the range spaces is the whole space, the strong convergence theorem can be proved by a very simple and elegant method found by N. Dunford. See section 9.4 for the case $\alpha = 0$.

In view of the importance of formula (9.3.10) for our theory we restate it explicitly in two different ways. The first formulation

$$T(\xi)x = \lim_{\eta \to 0} e^{-(\xi-\alpha)/\eta} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\xi - \alpha}{\eta}\right)^n T(\alpha + n\eta)x \tag{9.3.13}$$

shows that the result belongs to the theory of "singular series". The second form is more interesting: with the usual notation of the difference calculus

$$\Delta_\eta^n T(\alpha) = \eta^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} T(\alpha + k\eta), \tag{9.3.14}$$

we have

$$T(\xi)x = \lim_{\eta \to 0} \sum_{n=0}^{\infty} \frac{(\xi - \alpha)^n}{n!} \Delta_\eta^n T(\alpha)x. \tag{9.3.15}$$

This form shows that we are dealing with a generalization of Taylor's series.

We postpone further discussion of these formulas to Chapter XI where the infinitesimal generator will be studied in more detail. See also Chapter XVI for the applications to semi-groups of translations in which case the connection with the classical Taylor's series is the closest.

9.4. Continuity at the origin. So far α was supposed to be a positive number. The case of $\alpha = 0$, in many respects the most interesting, remains to be discussed.

THEOREM 9.4.1. *If $T(\xi)$ is defined for $\xi > 0$ and satisfies* (9.2.1) *and if*

$$\lim_{\eta \to 0} T(\eta) \equiv J \tag{9.4.1}$$

exists in the strong sense, then J is a projection operator, $J^2 = J$, which maps all of $\mathfrak{X}$ onto the closure of $\mathfrak{X}_0 = \bigcup_\alpha T(\alpha)[\mathfrak{X}]$, and

$$T(\xi) = JT(\xi) = T(\xi)J. \tag{9.4.2}$$

$T(\xi)$ is strongly continuous for $\xi \geqq 0$, if $T(0) = J$ by definition, and

$$T(\xi)x = \lim_{\eta \to 0} \exp [\xi A_\eta] Jx \tag{9.4.3}$$

for all x, the limit existing uniformly with respect to ξ in any finite interval $[0, \omega]$.

Finally, a set of necessary and sufficient conditions that the limit in (9.4.1) *shall exist with $J = I$ is that* (i) *$T(\xi)$ be strongly measurable for $\xi > 0$,* (ii) *there exists a finite positive M such that $\| T(\xi) \| \leqq M$ for $0 < \xi \leqq 1$, and* (iii) $\bar{\mathfrak{X}}_0 = \mathfrak{X}$.

PROOF. If $T(+0) = J$ exists as the strong limit of $T(\eta)$ we prove, using the same type of argument as in Theorem 8.4.1, successively that (1) $J^2 = J$; (2) $\| T(\xi) \| \leqq M(\omega)$ for $0 < \xi \leqq \omega < \infty$; (3) $\lim_{\eta \to 0} T(\xi + \eta) = T(\xi + 0)$ exists in the strong sense for $\xi \geqq 0$, uniformly with respect to ξ in $[0, \omega]$, and $T(\xi + 0) = JT(\xi) = T(\xi)J$; (4) $T(\xi + 0)$ satisfies (9.2.1) for $\xi \geqq 0$ and is strongly right-continuous, hence measurable and, by Theorem 9.2.1, actually strongly continuous for $\xi \geqq 0$; and (5) $T(\xi)$ is strongly continuous and $T(\xi + 0) \equiv T(\xi)$.

This proves (9.4.2) and shows that conditions (i) and (ii) are necessary for the existence of (9.4.1).

Since $Jx = \lim_{\eta \to 0} T(\eta)x$, it follows that $Jx \in \bar{\mathfrak{X}}_0$. On the other hand, if $x \in \bar{\mathfrak{X}}_0$, then $Jx = x$. Indeed, if $x \in \mathfrak{D}(A)$, then $\lim_{\eta \to 0} A_\eta x = Ax$ exists which requires that $\lim_{\eta \to 0} T(\eta)x = x$. Hence $Jx = x$ for $x \in \mathfrak{D}(A)$. By Theorem 9.3.1, $\mathfrak{D}(A)$ is dense in $\bar{\mathfrak{X}}_0$ so that $Jx = x$ everywhere in $\bar{\mathfrak{X}}_0$. Consequently $J[\mathfrak{X}] = \bar{\mathfrak{X}}_0$. It follows that (iii) is necessary in order that $J = I$.

The proof of (9.4.3) follows from formula (9.3.12) upon setting $\alpha = 0$, $T(0) = J$. We note that if $T(\xi)$ is uniformly continuous for $\xi > 0$ and not merely strongly continuous, then

$$\lim_{\eta \to 0} \| \exp [\xi A_\eta] J - T(\xi) \| = 0 \tag{9.4.4}$$

uniformly with respect to ξ in $0 < \epsilon \leqq \xi \leqq 1/\epsilon$.

For the special case in which $J = I$, N. Dunford has given a very simple and elegant proof of (9.4.3) (cf. N. Dunford and I. E. Segal [1]). He noticed that if $x \in \mathfrak{D}(A)$, then $\exp [(\xi - \tau)A_\eta]\, T(\tau)x$ is a differentiable function of τ (by virtue of formula (5.7.3) together with Theorem 9.3.2). Hence

$$\begin{aligned} T(\xi)x - \exp [\xi A_\eta]\, x &= \int_0^\xi \frac{d}{d\tau} \{\exp [(\xi - \tau)A_\eta]\, T(\tau)x\}\, d\tau \\ &= \int_0^\xi \exp [(\xi - \tau)A_\eta]\, T(\tau)[A_\eta x - Ax]\, d\tau. \end{aligned} \tag{9.4.5}$$

If $0 < \eta \leqq 1, M \geqq 1$, formula (9.3.9) shows that the last member is dominated in norm by

$$\| A_\eta x - Ax \| \int_0^\xi M^{1+\tau} e^{(M-1)(\xi - \tau)}\, d\tau.$$

This expression tends to zero with η, uniformly with respect to ξ in $[0, \omega]$ since the integral is uniformly bounded in this interval. It follows that (9.4.3) holds for $x \in \mathfrak{D}(A)$. But $\mathfrak{D}(A)$ is dense in $\mathfrak{X}$ and $\| \exp (\xi A_\eta) \|$ is uniformly bounded for $0 < \eta \leqq 1, 0 \leqq \xi \leqq \omega$, whence it follows that (9.4.3) holds for all x, uniformly with respect to ξ in $[0, \omega]$.

To prove the sufficiency of the conditions of Theorem 9.4.1 we note that (i) plus (ii) implies that $T(\xi)$ is strongly continuous for $\xi > 0$ and that $\mathfrak{D}(A)$ is dense in $\mathfrak{X}_0$ and by (iii) $\mathfrak{D}(A)$ is dense in $\mathfrak{X}$. But $\lim_{\eta \to 0} T(\eta)x = x$ in $\mathfrak{D}(A)$ and $\| T(\eta) \| \leqq M$ for $0 < \eta < 1$ by (ii), whence we conclude that the limit exists and equals x everywhere in $\mathfrak{X}$. This completes the proof.

Suppose now that $T(\xi)$ is strongly continuous for $\xi > 0$ and define an operator $T(0)$ by

$$\lim_{\eta \to 0} T(\eta)x = T(0)x \tag{9.4.6}$$

whenever the limit exists (in the strong sense). The domain of definition of $T(0)$ contains the closure of $\mathfrak{X}_0$ which we now denote by $\mathfrak{R}$. We also introduce

the linear space $\mathfrak{M}$ consisting of all x such that $T(0)x = \theta$. Here $\mathfrak{N}$ is closed by definition and a simple argument shows that $\mathfrak{M}$ is also closed. Further $\mathfrak{M}$ and $\mathfrak{N}$ have only the zero element in common. We can also characterize $\mathfrak{M}$ as the set of elements x which are annihilated by $T(\xi)$ for every $\xi > 0$. Indeed, if $T(\xi)x = \theta$ for all $\xi > 0$, then $T(0)x = \theta$ and $x \in \mathfrak{M}$. Conversely, if $x \in \mathfrak{M}$ then

$$\begin{aligned} T(\xi)x &= \lim_{\eta\to 0} T(\xi + \eta)x \\ &= \lim_{\eta\to 0} T(\xi)[T(\eta)x] = T(\xi)[T(0)x] = \theta, \end{aligned}$$

so that $T(\xi)x = \theta$ for every $x \in \mathfrak{M}$ and all $\xi > 0$. We also note that $T(0)x = x$ for $x \in \mathfrak{N}$.

THEOREM 9.4.2. *A necessary and sufficient condition that $\lim_{\eta\to 0} T(\eta)x$ shall exist in the strong sense for all x is that*

$$\mathfrak{X} = \mathfrak{M} \oplus \mathfrak{N}. \tag{9.4.7}$$

PROOF. (9.4.7) is an assertion that to every $x \in \mathfrak{X}$ there is a uniquely defined pair of elements x_1, x_2 such that

$$x = x_1 + x_2, \qquad x_1 \in \mathfrak{M},\ x_2 \in \mathfrak{N}.$$

If the condition holds then $T(0)x = x_2$ so that $T(0)$ is defined for all x and the condition is sufficient. Conversely, if $T(0)$ is defined for all x, then $T(0) = J$, a projection operator, and

$$x = Jx + (x - Jx)$$

is the required decomposition since $Jx \in \mathfrak{N}$, $x - Jx \in \mathfrak{M}$. The decomposition is obviously unique, so the condition is also necessary.

It should be observed that conditions (i) and (ii) of Theorem 9.4.1 are not sufficient to ensure the existence of $T(0)x$ for all x. This is shown by the following example (a power semi-group in the sense of section 16.7). We take $\mathfrak{X} = C[0, 1]$ and define

$$T(\xi)[x(t)] = \left[t\left(1 + \cos\frac{1}{t}\right)\right]^{\xi} x(t), \qquad \xi > 0.$$

This semi-group is strongly continuous for $\xi > 0$, but $T(0)[x(t)]$ exists if and only if $x(t) = 0$ at all the points t where the bracket equals zero, that is, for $t = 0$ and $1/[(2n + 1)\pi]$. The functions $x(t)$ having this property form the set $\mathfrak{X}_0 = \mathfrak{N}$ which is not dense in $\mathfrak{X}$ and $\mathfrak{M}$ contains only the zero-element so that (9.4.7) does not hold.

The form of Theorem 9.4.2 calls for additional comment. It is concerned with the decomposition of the given space $\mathfrak{X}$ into the direct sum of two subspaces, $\mathfrak{M}$ and $\mathfrak{N}$. Here $\mathfrak{N}$ is the closure of the union of the range spaces of the individual transformations of the semi-group and $\mathfrak{M}$ is the intersection of their "null" spaces. This is merely a special instance of a large class of decomposition theorems discovered by N. Dunford [9] who found that they govern a number of apparently unrelated questions such as ergodic theorems, Haar measure, invariant measure, Wiener's translation theorem, etc. We shall encounter

other such decomposition theorems in Chapter XIV which is concerned with ergodicity. Actually Theorem 9.4.2 is an ergodic theorem for a semi-group involving the limiting passage $\xi \to 0$ and the strong Cauchy limit.

2. Extensions

9.5. Problem B in the strong topology. We now consider two complex (B)-spaces $\mathfrak{X}$ and $\mathfrak{Y}$ and a function $T(x)$ on $\mathfrak{X}$ to $\mathfrak{E}(\mathfrak{Y})$ satisfying the following conditions:

(1) *$T(x)$ is defined for $x \in \mathfrak{K}$, an open positive cone in the sense of Definition 8.5.1, and has values in $\mathfrak{E}(\mathfrak{Y})$, that is, $T(x)$ is a bounded linear transformation on $\mathfrak{Y}$ to itself for fixed $x \in \mathfrak{K}$.*

(2) *There exists a finite positive M such that for all x in $\mathfrak{K}$ with $\|x\| \leqq 1$ we have $\|T(x)\| \leqq M$.*

(3) *$T(\xi x)y$ is a measurable function of ξ for $\xi > 0$, $x \in \mathfrak{K}$, $y \in \mathfrak{Y}$.*

(4) *For $x_1, x_2 \in \mathfrak{K}$, $y \in \mathfrak{Y}$,*

$$T(x_1 + x_2)[y] = T(x_1)\{T(x_2)[y]\}. \tag{9.5.1}$$

Thus the operators $\{T(x)\}$, $x \in \mathfrak{K}$, form a semi-group $\mathfrak{S}$ of linear bounded transformations, the parameter manifold being the cone $\mathfrak{K}$ of $\mathfrak{X}$ which is a semi-module with operators, the multipliers being all positive numbers. We assume that $\mathfrak{K} \neq \mathfrak{X}$.

Theorem 9.5.1. *$T(\xi x)y$ is a continuous function of ξ for $\xi > 0$ and fixed $x \in \mathfrak{K}$, $y \in \mathfrak{Y}$. More generally, if $\mathfrak{X}_{(n)}$ is a finite-dimensional linear subspace of $\mathfrak{X}$, then $T(x)y$ is a continuous function of x in $\mathfrak{X}_{(n)} \cap \mathfrak{K}$. In particular, if $\mathfrak{K}$ itself is finite-dimensional, then $T(x)y$ is continuous for x in $\mathfrak{K}$.*

The first assertion follows from Theorem 9.2.1 by virtue of conditions (2) and (3). The proof of the second one follows the same pattern as that of Theorems 8.5.1 and 9.2.1 and the details are left to the reader.

For $x \in \mathfrak{K}$ we now define

$$A_\eta(x)y = \frac{1}{\eta}[T(\eta x) - I]y, \tag{9.5.2}$$

$$A(x)y = \lim_{\eta \to 0} A_\eta(x)y \tag{9.5.3}$$

whenever the limit exists. The domain and range of $A(x)$ will be denoted by $\mathfrak{D}[A(x)]$ and $\mathfrak{R}[A(x)]$ respectively. We also introduce the range of $T(\alpha x)$ denoted by $\mathfrak{R}[T(\alpha x)] = \mathfrak{Y}_\alpha(x)$ and put $\mathfrak{Y}_0(x) = \bigcup_\alpha \mathfrak{Y}_\alpha(x)$. The next two theorems will elucidate the relations between these linear subspaces of $\mathfrak{Y}$. In the first theorem comparisons are made for fixed x, in the second one for different values of x.

THEOREM 9.5.2. *For fixed x in $\mathfrak{R}$, $\mathfrak{D}[A(x)]$ is dense in $\mathfrak{Y}_0(x)$ and the two spaces have the same closure. $\mathfrak{R}[A(x)]$ is also contained in the closure of $\mathfrak{Y}_0(x)$. If y belongs to the closure of $\mathfrak{Y}_0(x)$, then $\lim_{\eta\to 0} T(\eta x)y = y$.*

PROOF. For fixed x in $\mathfrak{R}$, $T(\xi x)$ satisfies the conditions of Theorem 9.3.1 and the first three assertions follow from this theorem. By reason of condition (2), the relation $\lim_{\eta\to 0} T(\eta x)y = y$, which obviously holds in $\mathfrak{D}[A(x)]$, must also hold in the closure of this space. This is the last assertion and completes the proof.

THEOREM 9.5.3. *For every $\alpha > 0$ we have $\mathfrak{D}[A(\alpha x)] = \mathfrak{D}[A(x)]$. Further $\mathfrak{D}[A(x_1 + x_2)] \supset \mathfrak{D}[A(x_1)] \cap \mathfrak{D}[A(x_2)]$ and $\mathfrak{D}[A(x_1)A(x_2)]$ is dense in the intersection. For y in the latter,*

$$\lim_{\eta\to 0} T(\eta x_1)A(x_2)y = A(x_2)y. \tag{9.5.4}$$

PROOF. The first relation is a trivial consequence of

$$\lim_{\eta\to 0} \frac{1}{\eta\alpha}[T(\eta\alpha x) - I]y = \lim_{\eta\to 0} \frac{1}{\eta}[T(\eta x) - I]y \tag{9.5.5}$$

when either limit exists. Suppose now that $y \in \mathfrak{D}[A(x_1)] \cap \mathfrak{D}[A(x_2)]$. Then $T(\beta x_2)y \in \mathfrak{D}[A(x_1)]$, hence also $A_\beta(x_2)y$; the latter expression tends to a limit when $\beta \to 0$ since $y \in \mathfrak{D}[A(x_2)]$. Thus $A(x_2)y$ is in the closure of $\mathfrak{D}[A(x_1)]$ and (9.5.4) holds by the preceding theorem. We may of course interchange x_1 and x_2 in this relation.

Next if $y \in \mathfrak{D}[A(x_1)] \cap \mathfrak{D}[A(x_2)]$ we note that $\lim_{\alpha\to 0,\beta\to 0} T(\alpha x_1 + \beta x_2)y = y$. Forming

$$y_n = n^2 \int_\alpha^{\alpha+1/n} \int_\beta^{\beta+1/n} T(\xi_1 x_1 + \xi_2 x_2)y\, d\xi_2\, d\xi_1,$$

one finds that

$$A(x_1)A(x_2)y_n = n^2\left[T\left(\frac{1}{n}x_1\right) - I\right]\left[T\left(\frac{1}{n}x_2\right) - I\right]T(\alpha x_1 + \beta x_2)y$$

and $\lim_{n\to\infty} y_n = T(\alpha x_1 + \beta x_2)y$. This implies that the domain of $A(x_1)A(x_2)$ is dense in the intersection of the domains of $A(x_1)$ and $A(x_2)$. Finally we shall show that if y belongs to this intersection then $A(x_1 + x_2)y$ exists. Indeed,

$$A_\eta(x_1 + x_2)y = T(\eta x_1)A_\eta(x_2)y + A_\eta(x_1)y.$$

Here the second term clearly tends to $A(x_1)y$ when $\eta \to 0$ and the first term equals

$$T(\eta x_1)[A_\eta(x_2)y - A(x_2)y] + T(\eta x_1)A(x_2)y.$$

In this expression the first term tends to θ when $\eta \to 0$, since $\| T(\eta x_1) \| \leqq M$ for $0 < \eta \leqq 1/\| x_1 \|$ and the expression in square brackets tends to θ when $\eta \to 0$. It follows then from (9.5.4) that $A(x_1 + x_2)y$ exists and

$$(9.5.6) \qquad A(x_1 + x_2)y = A(x_1)y + A(x_2)y.$$

In the same manner we prove

THEOREM 9.5.4. *$\mathfrak{D}[A(x_1)A(x_2) \cdots A(x_k)]$ is dense in $\bigcap_1^k \mathfrak{D}[A(x_m)]$. In particular, $\mathfrak{D}\{[A(x)]^k\}$ is dense in $\mathfrak{D}[A(x)]$.*

From formulas (9.5.5) and (9.5.6) the following important result is obtained:

THEOREM 9.5.5. *$A(x)$ is an additive, positive-homogeneous function of x. More precisely expressed,* (i) *if $A(x_1)y$ and $A(x_2)y$ exist so does $A(x_1 + x_2)y$ and is given by* (9.5.6), *and* (ii) *if $A(x)y$ exists so does $A(\alpha x)y$ and $A(\alpha x)y = \alpha A(x)y$, $\alpha > 0$.*

Finally we have the following analog of Theorem 9.4.1.

THEOREM 9.5.6. *If*

$$(9.5.7) \qquad \lim_{\eta \to 0} T(\eta x)y$$

exists for all $x \in \mathfrak{R}$, $y \in \mathfrak{Y}$, then the limit is a projection operator J independent of x, $J^2 = J$, which maps all of $\mathfrak{Y}$ upon a closed subspace $\mathfrak{Y}_0$. Every space $\mathfrak{Y}_0(x)$, $x \in \mathfrak{R}$, has $\mathfrak{Y}_0$ as its closure. Further

$$(9.5.8) \qquad T(x) = JT(x) = T(x)J.$$

$T(x)$ is strongly continuous as a function of x in $\mathfrak{R} \cap \mathfrak{X}_{(n)}$ where $\mathfrak{X}_{(n)}$ is any finite-dimensional subspace of $\mathfrak{X}$, and

$$(9.5.9) \qquad T(x)y = \lim_{\eta \to 0} \exp\,[A_\eta(x)]\,Jy$$

for all y, the limit existing uniformly with respect to x in any bounded portion of $\mathfrak{R} \cap \mathfrak{X}_{(n)}$.

Finally, a sufficient set of conditions in order that (9.5.7) *shall exist with $J = I$ is that* (1)–(4) *hold and, in addition,* (5) *$\mathfrak{Y}_0(x)$ is dense in $\mathfrak{Y}$ for each x.*

PROOF. Denoting temporarily the limit in (9.5.7) by $J(x)$ we obtain from Theorem 9.4.1 that $J(x)$ is a projection operator, $[J(x)]^2 = J(x)$, which maps all of $\mathfrak{Y}$ on the closure of $\mathfrak{Y}_0(x)$, and $T(\xi x) = J(x)T(\xi x) = T(\xi x)J(x)$ for $\xi > 0$. Further $T(\xi x)$ is a strongly continuous function of ξ for $\xi > 0$, and

$$(9.5.10) \qquad T(\xi x)y = \lim_{\eta \to 0} \exp\,[\xi A_\eta(x)]\,J(x)y$$

for all y, the limit existing uniformly with respect to ξ in any finite interval $[0, \omega]$. We note next that if $\mathfrak{X}_{(n)}$ is any finite-dimensional linear subspace of $\mathfrak{X}$, then by Theorem 9.5.1 $T(x)$ is strongly continuous for $x \in \mathfrak{R} \cap \mathfrak{X}_{(n)}$.

To prove that $J(x)$ is independent of x, we argue as in the proof of Theorem 8.5.2. From

$$T(\alpha x_1 + \beta x_2)y = T(\alpha x_1)T(\beta x_2)y = T(\beta x_2)T(\alpha x_1)y$$

we get, letting $\alpha \to 0$ and using continuity on finite-dimensional subspaces,

$$T(\beta x_2)y = J(x_1)T(\beta x_2)y = T(\beta x_2)J(x_1)y.$$

When $\beta \to 0$ one obtains

$$J(x_2)y = J(x_1)J(x_2)y = J(x_2)J(x_1)y$$

and, interchanging x_1 and x_2, this gives $J(x_1)y = J(x_2)y$ whence $J(x_1) = J(x_2) = J$ for all x_1, x_2. Since J maps $\mathfrak{Y}$ on the closure of $\mathfrak{Y}_0(x)$ for every x in $\mathfrak{R}$, this closure must be independent of x.

Formula (9.5.9) now follows from (9.5.10). The uniformity with respect to x for $x \in \mathfrak{R} \cap \mathfrak{X}_{(n)}$, $\|x\| \leqq \omega$, follows from the estimate

$$\| \exp [A_\eta(x)] Jy - T(x)y \| \leqq \mu(\delta, y) + \eta\delta^{-2}M^{1+\omega}[1 + 2e^{M\omega}] \| y \|,$$

valid for $0 < \eta \leqq \eta_0 = \eta_0(\delta, \omega, M)$, where $\mu(\delta, y)$ is the rectified modulus of continuity of $T(x)y$ in the domain in question. The estimate is obtained in the same manner as (9.3.12).

It is obvious from Theorem 9.4.1 and the preceding discussion that conditions (1)–(5) are sufficient in order that the limit in (9.5.7) shall exist for all x in $\mathfrak{R}$ and have the value I. On the other hand, if the limit does exist and equals I for all x in $\mathfrak{R}$ and (4) holds, that is, we are dealing with a semi-group, then conditions (1), (3), and (5) are seen to hold so they are also necessary. Condition (2) is of a different nature; all we can assert here is that (2) is necessary if the limit is to hold uniformly with respect to x in the unit sphere.

9.6. The set of generators. From the preceding discussion we see that with the semi-group $\mathfrak{S} = \{T(x)\}$ of linear bounded transformations on $\mathfrak{Y}$ to itself there is associated the set $\mathfrak{A} = \{A(x)\}$ of infinitesimal generators. Here $A(x)$ is also a linear transformation on $\mathfrak{Y}$ to itself, but ordinarily $A(x)$ is not bounded and hence not an element of $\mathfrak{E}(\mathfrak{Y})$. While the set $\mathfrak{S}$ is multiplicative, $\mathfrak{A}$ is additive and corresponds to the *Lie ring* of a continuous group. If $A_1, A_2 \in \mathfrak{A}$ so does $\alpha_1 A_1 + \alpha_2 A_2$ when $\alpha_1 \geqq 0$, $\alpha_2 \geqq 0$, that is, $\mathfrak{A}$ *is a semi-module with operators, the multipliers being positive numbers.* Thus there is a marked difference between our set $\mathfrak{A}$ and a Lie ring, the latter being a module admitting all real numbers as multipliers. This difference of course goes back to the basic difference between a group and a semi-group.

In a Lie ring there is also defined a notion of multiplication, the *commutator* $UV - VU$ being regarded as the product of U and V. In our case it is a priori plausible that the elements of $\mathfrak{A}$ commute, but owing to the unbounded character of $A(x)$, the proof of this fact requires a fairly elaborate argument in the course of which we shall establish several other important properties of $A(x)$.

THEOREM 9.6.1. *The operators $T(x_1)$ and $A(x_2)$ commute in the sense that $T(x_1)A(x_2)y = A(x_2)T(x_1)y$ provided $y \in \mathfrak{D}[A(x_2)]$.*

PROOF. We have

$$T(x_1)A_\eta(x_2)y = A_\eta(x_2)T(x_1)y.$$

If $y \in \mathfrak{D}[A(x_2)]$, the left side tends to $T(x_1)A(x_2)y$ when $\eta \to 0$. It follows that the right member has a limit which, by definition, is $A(x_2)T(x_1)y$ and that the limits are equal.

The next two theorems are due to I. Gelfand [3] for one-parameter groups but his proofs carry over to the present situation.

THEOREM 9.6.2. *If $y \in \mathfrak{D}[A(x)]$ and $\xi > 0$, then*

$$\frac{d}{d\xi} T(\xi x)y = A(x)T(\xi x)y = T(\xi x)A(x)y. \tag{9.6.1}$$

If y is in the closure of $\mathfrak{D}[A(x)]$ and $\alpha > 0$, then

$$A(x)\int_0^\alpha T(\xi x)y\,d\xi = [T(\alpha x) - I]y. \tag{9.6.2}$$

PROOF. In the first case

$$\frac{1}{\eta}[T((\xi + \eta)x) - T(\xi x)]y = A_\eta(x)T(\xi x)y = T(\xi x)A_\eta(x)y$$

where by assumption the last member tends to the limit $T(\xi x)A(x)y$ when $\eta \to 0$, thus proving (9.6.1). Further

$$A_\eta(x)\int_0^\alpha T(\xi x)y\,d\xi = \frac{1}{\eta}\int_\alpha^{\alpha+\eta} T(\tau x)y\,d\tau - \frac{1}{\eta}\int_0^\eta T(\tau x)y\,d\tau.$$

When $\eta \to 0$ the first expression on the right tends to $T(\alpha x)y$ since $T(\xi x)y$ is a continuous function of ξ at $\xi = \alpha$ for any y, while the second tends to y provided y is in the closure of $\mathfrak{D}[A(x)]$. This completes the proof.

If $y \in \mathfrak{D}[A(x)]$ we may pass to the limit under the sign of integration obtaining

$$\int_0^\alpha T(\xi x)A(x)y\,d\xi = [T(\alpha x) - I]y. \tag{9.6.3}$$

These formulas have analogs for functions of several variables which may be obtained by induction from the one variable case. In deriving the formulas, attention should be paid to the order of the limiting passages and to successive restrictions imposed upon the element y. Thus if $y \in \mathfrak{D}[A(x_1) \cdots A(x_k)]$

$$\frac{\partial^k}{\partial\xi_1 \cdots \partial\xi_k} T(\xi_1 x_1 + \cdots + \xi_k x_k)y = T(\xi_1 x_1 + \cdots + \xi_k x_k)A(x_1)\cdots A(x_k)y. \tag{9.6.4}$$

If y is in the intersection of the closures of $\mathfrak{D}[A(x_1)], \cdots, \mathfrak{D}[A(x_k)]$ we have

$$A(x_1)\cdots A(x_k)\int_0^{\alpha_1}\cdots\int_0^{\alpha_k} T(\xi_1 x_1 + \cdots + \xi_k x_k)y\,d\xi_1\cdots d\xi_k = \prod_{j=1}^{k}[T(\alpha_j x_j) - I]y \tag{9.6.5}$$

where evidently the order of the unbounded operators is immaterial. If in addition $y \in \mathfrak{D}[A(x_1) \cdots A(x_k)]$ the operators $A(x_j)$ can be taken inside the integral, starting with $A(x_k)$ and ending with $A(x_1)$, so that

$$(9.6.6)\quad \int_0^{\alpha_1} \cdots \int_0^{\alpha_k} T(\xi_1 x_1 + \cdots + \xi_k x_k) A(x_1) \cdots A(x_k) y \, d\xi_k \cdots d\xi_1 = \prod_{j=1}^{k} [T(\alpha_j x_j) - I] y,$$

order being essential.

THEOREM 9.6.3. *The operator* $A(x_1) \cdots A(x_k)$ *is closed for any choice of* k *and* $x_1, \cdots, x_k$ *in* $\mathfrak{R}$.

PROOF. We start with $k = 1$. Suppose that $\{y_n\}$ is a sequence of elements in $\mathfrak{D}[A(x)]$ and that $y_n \to y_0$, $A(x)y_n \to z_0$. It is required to prove that $A(x)y_0$ exists and equals z_0. Formula (9.6.3) holds for $y = y_n$ so that

$$\int_0^{\alpha} T(\xi x) A(x) y_n \, d\xi = [T(\alpha x) - I] y_n.$$

Passing to the limit with n one obtains

$$\frac{1}{\alpha} \int_0^{\alpha} T(\xi x) z_0 \, d\xi = \frac{1}{\alpha} [T(\alpha x) - I] y_0.$$

Here z_0 is in the closure of $\mathfrak{D}[A(x)]$ so the left side tends to z_0 when $\alpha \to 0$. Hence $A(x)y_0$ exists and equals z_0.

We note next that $T(x)$, being a bounded everywhere defined transformation, is closed. Hence if U is a closed transformation on $\mathfrak{Y}$ to itself, $UT(x)$ is also closed and so is $T(x)U$ if U and $T(x)$ commute.

Suppose now that we have proved the closure of any product of generators involving not more than $(k - 1)$ factors. We shall show that $A(x_1) \cdots A(x_k)$ is also closed. Suppose that $\{y_n\}$ is a sequence in $\mathfrak{D}[A(x_1) \cdots A(x_k)]$ and that $y_n \to y_0$, $A(x_1) \cdots A(x_k)y_n \to z_0$. We note that $A(x_2) \cdots A(x_k)y_n \in \mathfrak{D}[A(x_1)]$ for every n. Hence

$$\int_0^{\alpha} T(\xi x_1) A(x_1) A(x_2) \cdots A(x_k) y_n \, d\xi = [T(\alpha x_1) - I] A(x_2) \cdots A(x_k) y_n.$$

Here the operator on the right is closed and the expression on the left tends to a limit when $n \to \infty$. It follows that

$$\frac{1}{\alpha} \int_0^{\alpha} T(\xi x_1) z_0 \, d\xi = \frac{1}{\alpha} [T(\alpha x_1) - I] A(x_2) \cdots A(x_k) y_0.$$

Passing to the limit with α and remembering that z_0 is in the closure of $\mathfrak{D}[A(x_1)]$ we get $z_0 = A(x_1) \cdots A(x_k)y_0$. This completes the induction proof.

We come now to the main result:

THEOREM 9.6.4. *If* y *is in the intersection of* $\mathfrak{D}[A(x_1)]$ *and* $\mathfrak{D}[A(x_1)A(x_2)]$ *then it is also in* $\mathfrak{D}[A(x_2)A(x_1)]$ *and*

$$A(x_1)A(x_2)y = A(x_2)A(x_1)y.$$

PROOF. We have

$$A(x_1)A(x_2)y = \lim_{\eta\to 0} A_\eta(x_1)A(x_2)y$$
$$= \lim_{\eta\to 0} A(x_2)A_\eta(x_1)y = A(x_2)A(x_1)y$$

where the last step follows from the fact that (i) $\lim_{\eta\to 0} A_\eta(x_1)y$ exists and equals $A(x_1)y$ and (ii) $A(x_2)$ is closed.

The assumption that $y \in \mathfrak{D}[A(x_1)]$ is essential and is not necessarily implied by the existence of $A(x_1)A(x_2)y$. The following counter example may be helpful in clarifying the situation. It is open to the objection that the corresponding set $\mathfrak{R}$ is not open, but this situation has to be faced in the next section anyway. We take $\mathfrak{Y} = C[0, \infty]$ with the usual metric and define a two-parameter semi-group by

$$T(\xi_1, \xi_2)[y(t)] = \exp[-\xi_1 t - \xi_2/t]y(t)$$

for $\xi_1 \geqq 0$, $\xi_2 \geqq 0$, $(\xi_1, \xi_2) \neq (0, 0)$, the value of the transform for $t = 0$ being defined by continuity. Here there are two infinitesimal generators A_1 and A_2 where

$$A_1[y(t)] = -ty(t), \qquad A_2[y(t)] = -(1/t)y(t).$$

$\mathfrak{D}[A_1]$ is the subset of $C[0, \infty]$ on which $ty(t)$ tends to a finite limit when $t \to \infty$, while $\mathfrak{D}[A_2]$ is the subset on which $y(t)/t$ tends to a finite limit when $t \to 0$. Here we have $\mathfrak{D}[A_1A_2] = \mathfrak{D}[A_2]$ and $\mathfrak{D}[A_2A_1] = \mathfrak{D}[A_1]$. On the former set $A_1A_2 = I$, on the latter $A_2A_1 = I$, but $\mathfrak{D}[A_1] \neq \mathfrak{D}[A_2]$.

The problem of permuting three or more generators can be handled in a similar manner. For cylic permutation of k generators it is sufficient that

$$y \in \bigcap_{j=1}^{k-1} \mathfrak{D}[A(x_j)] \cap \mathfrak{D}[A(x_1)\cdots A(x_k)],$$

but if general permutations are to be allowed the conditions become fairly complicated.

9.7. The n-parameter semi-groups. The most important application of the preceding theory appears to be to the case of n-parameter semi-groups of linear bounded transformations of the type considered in ergodic theory. Here $\mathfrak{X} = E_n$ is a real euclidean space of n dimensions with the usual definitions of arithmetical operations and metric. We write $x = (\xi_1, \cdots, \xi_n)$ and denote the unit vectors by $u_1, \cdots, u_n$ where $u_j = (\delta_{jk})$. The assumptions (1)–(4) of section 9.5 will now be replaced by

(i) *$T(x)$ is defined for $x \in E_n^+$, the 2^n-ant in which $\xi_1 \geqq 0, \cdots, \xi_n \geqq 0$, excluding $(0, \cdots, 0)$, and has values in $\mathfrak{E}(\mathfrak{Y})$.*

(ii) *and* (iv) = (2) *and* (4) *with $\mathfrak{R}$ replaced by E_n^+.*

(iii) *$T(x)y$ is a measurable function of x in E_n^+ for every fixed $y \in \mathfrak{Y}$.*

The set E_n^+ is a positive cone but not open; (iii) is evidently much less restrictive than (3). Nevertheless the following result holds.

THEOREM 9.7.1. *$T(x)y$ is a continuous function of x in the interior of E_n^+ for each y in $\mathfrak{Y}$.*

The proof follows the same lines as the proofs of Theorems 8.5.1 and 9.5.1, the only difference being that now the measurability of $T(x)y = T(\sum_1^n \xi_k u_k)y$ as a function of $(\xi_1, \cdots, \xi_n)$ is postulated. The details are left to the reader.

In view of this situation most of the results of sections 9.5 and 9.6 apply to the present case. We get a simpler and more satisfactory theory, however, by adding a fifth assumption:

(v) $\mathfrak{Y}_0(u_k)$ *is dense in* $\mathfrak{Y}$ *for* $k = 1, 2, \cdots, n$.

We recall that $\mathfrak{Y}_0(u_k)$ is the union of the range spaces of $T(\alpha u_k)$ for $\alpha > 0$. We can then state a stronger result:

THEOREM 9.7.2. *Let* $\mathfrak{S} = \{T(x)\} = \{T(\xi_1, \cdots, \xi_n)\}$ *be an n-parameter semi-group satisfying assumptions* (i)–(v). *Then*

(1) $T(\xi_1, \cdots, \xi_n)$ *is strongly continuous in* E_n^+ *and tends strongly to the identity when* $(\xi_1, \cdots, \xi_n) \to (0, \cdots, 0)$;

(2) $T(\xi_1, \cdots, \xi_n)$ *is the direct product of n continuous one-parameter semi-groups* $\mathfrak{S}_k = \{T_k(\xi_k)\} = \{T(\xi_k u_k)\}$ *so that*

$$T(\xi_1, \cdots, \xi_n) = \prod_1^n T_k(\xi_k);$$

(3) $T_j(\xi_j)$ *commutes with* $T_k(\xi_k)$ *and* $\lim_{\eta \to 0} T_k(\eta)y = y$;

(4) $T_k(\xi_k)$ *is generated by the infinitesimal transformation* $A_k = A(u_k)$ *and all the generators of* $\mathfrak{S}$ *are of the form* $A = \sum_1^n \xi_k A_k$, $\xi_k \geqq 0$.

REMARK. We have assumed that all the parameters are essential. If this is not the case, the theorem is still valid but the basic generators A_k are no longer linearly independent.

PROOF. We know already that $T(\xi_1, \cdots, \xi_n)$ is continuous in Int (E_n^+), but it remains to prove continuity on the boundary. The first step is to prove that $\mathfrak{Y}_0(x)$ is dense in $\mathfrak{Y}$ for every $x \in$ Int (E_n^+). This follows from (ii) + (v) by the following argument. Let y_0 be a fixed element of $\mathfrak{Y}$. To any $\epsilon > 0$ we can find elements $y_1, \cdots, y_n$ in $\mathfrak{Y}$ and positive numbers $\xi_{01}, \cdots, \xi_{0n}$ such that

$$\| y_{k-1} - T(\xi_{0k} u_k) y_k \| \leqq \epsilon, \qquad k = 1, \cdots, n.$$

If $\sum_1^n \xi_{0k} u_k = x_0$, we conclude that

$$\| y_0 - T(x_0) y_n \| \leqq \epsilon \frac{M^n - 1}{M - 1} = \epsilon_1 .$$

If $x \in$ Int (E_n^+) is given we can choose α, $\alpha > 0$, so small that $x_0 - \alpha x \in$ Int (E_n^+). Hence

$$\| y_0 - T(\alpha x)[T(x_0 - \alpha x) y_n] \| \leqq \epsilon_1 ,$$

so that $\mathfrak{Y}_0(x)$ is dense in $\mathfrak{Y}$. From this it follows that the conditions of Theorem 9.5.6 are satisfied and $\lim_{\eta \to 0} T(\eta x) y = y$ for every $x \in$ Int (E_n^+), $y \in \mathfrak{Y}$. In particular

$$\lim_{\eta \to 0} T(\xi_k u_k + \eta x) y = T(\xi_k u_k) y. \tag{9.7.1}$$

Here $\xi_k u_k + \eta x \in$ Int (E_n^+) so that $T(\xi_k u_k + \eta x) y$ is a continuous function of ξ_k for $\xi_k > 0$ and fixed $\eta > 0$. It follows that $T(\xi_k u_k) y$, being the limit of a

sequence of continuous functions of ξ_k, is measurable in ξ_k. Further $\| T(\xi_k u_k) \| \leqq M$ for $0 < \xi_k \leqq 1$. By Theorem 9.2.1, $T(\xi_k u_k)y$ is then a continuous function of ξ_k for $\xi_k > 0$, $k = 1, \cdots, n$, and, by Theorem 9.4.1, $\lim_{\eta \to 0} T(\eta u_k)y = y$. Since

$$T(x)y = T\left(\sum_1^n \xi_k u_k\right) y = \prod_1^n T(\xi_k u_k)y \equiv \prod_1^n T_k(\xi_k)y$$

we conclude that $T(x)y$ is continuous in E_n^+. This proves (1)–(3).

The set of generators of $\mathfrak{S}$ is $\mathfrak{A} = \{A(x)\}$. By Theorem 9.5.5

$$A(x) = A\left(\sum_1^n \xi_k u_k\right) = \sum_1^n \xi_k A(u_k) \equiv \sum_1^n \xi_k A_k .$$

Here A_k is clearly the infinitesimal generator of $T_k(\xi_k)$. This completes the proof. For this theorem see also the recent paper of N. Dunford and I. E. Segal [1].

Assumption (v) is essential for the validity of this result. If it does not hold a peculiar situation will arise which we shall illustrate in the simplest case $n = 2$. We take two continuous one-parameter groups $\mathfrak{S}_1 = \{T_1(\xi)\}$, $\mathfrak{S}_2 = \{T_2(\eta)\}$ of linear bounded transformations on $\mathfrak{Y}$ to itself. We assume that the elements of $\mathfrak{S}_1$ commute with those of $\mathfrak{S}_2$ and that $\lim_{\xi \to 0} T_1(\xi) = J_1$, $\lim_{\eta \to 0} T_2(\eta) = J_2$ where $J_1 J_2 = J_2 J_1 \neq J_1$ and J_2. Then $\{T_1(\xi)T_2(\eta)\}$ is a two-parameter semi-group $\mathfrak{S}$ defined for $\xi \geqq 0, \eta \geqq 0, (\xi, \eta) \neq (0, 0)$ and satisfying conditions (i)–(iv) but not (v). Here $T_1(\xi)T_2(\eta)$ is continuous in the interior of the first quadrant and tends to continuous boundary values on the axes. On the ξ-axis the limit is $J_2 T_1(\xi) \neq T_1(\xi)$, on the η-axis it is $J_1 T_2(\eta) \neq T_2(\eta)$.

9.8. Remarks on Problem C in the strong topology. The extension of the results of §8.3 to the strong topology meets with considerable difficulties which have not been overcome at the time of the present writing. We shall therefore restrict ourselves to brief indications of what may be extended and where the difficulties set in.

We assume $G(\alpha, \beta)$ as usual to be a symmetric numerically-valued function of the two complex variables α and β holomorphic in some domain Δ of the complex plane. Further the addition formula

$$f(\zeta_1 + \zeta_2) = G[f(\zeta_1), f(\zeta_2)]$$

shall have an analytic solution $\varphi(\zeta)$, holomorphic in $|\zeta| < \rho$ and having its values in Δ. Moreover $\varphi(\zeta)$ shall not be a constant.

Theorem 8.6.1 admits of the following extension, the proof of which is left to the reader.

THEOREM 9.8.1. *Let $T(\xi)$ be a strongly measurable function of ξ on the open interval $(0, \omega)$ to $\mathfrak{E}(\mathfrak{X})$ such that*

$$T(\xi_1 + \xi_2)x = G[T(\xi_1), T(\xi_2)]x$$

for all $x \in \mathfrak{X}$ and all ξ_1 and ξ_2 in the interval and such that $T(\xi_1)$ and $T(\xi_2)$ commute. Let $\mathfrak{R}$ be the range of $T(\xi)$ in $\mathfrak{E}(\mathfrak{X})$ for $0 < \xi < \omega$. Suppose that $\mathfrak{R}$ is bounded and $\mathfrak{R} \subset \mathfrak{D}(\Delta_0)$, $\bar{\Delta}_0 \subset \Delta$. Suppose further that to every x there is a finite $M(x)$ such that for all ξ_1, ξ_2, η in $(0, \omega)$

$$\| G[T(\xi_1), T(\eta)]x - G[T(\xi_2), T(\eta)]x \| \leqq M(x) \, \| [T(\xi_1) - T(\xi_2)]x \|.$$

Then $T(\xi)$ is strongly continuous in $(0, \omega)$.

The justification of the Lipschitz condition is trivial if $G(\alpha, \beta)$ is entire; in other cases the situation is not so favorable. The crux of the matter is the extension of Lemma 8.7.1 to the strong topology.

The analog of the infinitesimal generator of a semi-group is given by the operator A defined as the strong limit of A_η when $\eta \to 0$ where

$$A_\eta = \frac{1}{\eta}[T(\eta) - \varphi(0)I].$$

It is likely that the domain of the operator A contains all elements of the form

$$\int_\alpha^\beta G_1[\varphi(0)I, T(\xi)]x\, d\xi, \qquad 0 < \alpha < \beta < \omega, \qquad x \in \mathfrak{X}.$$

and that these elements are dense in $\mathfrak{X}$ if $\bigcup_\alpha T(\alpha)[\mathfrak{X}]$ has this property.

A still more complicated problem is to decide whether or not the operator $\varphi(\xi A_\eta)$ has a sense and tends strongly to $T(\xi)$ when $\eta \to 0$. This is, of course, what one would expect by analogy with the semi-group case but we really have no evidence one way or the other.

CHAPTER X

LAPLACE INTEGRALS AND BINOMIAL SERIES

10.1 Orientation. An important chapter in classical analysis is concerned with functions which are holomorphic in a half-plane. The point at infinity is usually a singular point of such a function and the rate of growth of the function on rays or on vertical lines gives useful information concerning the behavior of the function in the neighborhood of the singularity. The rate of growth is also decisive for the representation problem. Here we have several possibilities; the function may be representable by a Cauchy or Poisson integral in terms of the boundary values on the line bounding the half-plane, or by one of the several forms of the Laplace integral, or by a suitable interpolation series such as the binomial series, to mention just a few of the alternatives. The rate of growth of the function decides what representations are possible.

In the theory of semi-groups of linear bounded transformations we are to be concerned with several vector-valued functions which are holomorphic in a half-plane. The resolvent $R(\lambda; A)$ of the infinitesimal generator is such a function and in some of the most important cases the semi-group operator itself has the same property. For an effective study of these functions we are forced to carry over the classical theory to vector-valued functions. The results will be used steadily in the following and are indispensable for Chapters XI to XV.

The present chapter is divided into three paragraphs: *Laplace transforms*, *Functions Holomorphic in a Half-Plane*, and *Binomial Series*. In the first we develop the elements of the theory of vector-valued Laplace-Stieltjes integrals, including the theory of convergence, analytic properties, scalar multiplication, and inversion formulas. For the classical theory we refer the reader to D. V. Widder's excellent monograph [1]. The second paragraph is concerned with functions of class $H_p(\alpha; \mathfrak{X})$, certain concepts of order and the associated growth measuring functions, and the problem of representing functions by Laplace integrals. For the classical theory the reader may consult E. Hille [3] and E. Hille and J. D. Tamarkin [5, 6] where further references are to be found. The theory of binomial series is equivalent to the theory of functions which are of exponential type in a half-plane. Here the classical theory is due essentially to F. Carlson [2] and N. E. Nörlund [2] to whose writings we refer for further details. An extension of this theory to vector-valued functions was given by the present author in [7]; the present account, though sketchy, has certain advantages.

References. Carlson [2], Hille [3, 7], Hille and Tamarkin [5, 6], Lindelöf [2], Nörlund [2], Phragmén [1], Phragmén and Lindelöf [1], Post [1], Widder [1].

1. Laplace Transforms

10.2. Laplace-Stieltjes integrals. Let $\mathfrak{X}$ be a complex (B)-space, $a(\xi)$ a function on $[0, \infty)$ to $\mathfrak{X}$, and let $a(\xi)$ be of *strongly bounded variation* over every finite interval $[0, \omega]$ in the sense of Definition 3.4.4 (3). Since a function of strongly bounded variation has right and left limits everywhere and is continuous except for a countable set of discontinuities of the first kind, we may normalize $a(\xi)$ by assuming that

$$a(0) = \theta, \qquad a(\xi) = \tfrac{1}{2}[a(\xi - 0) + a(\xi + 0)] \text{ for } \xi > 0.$$

The integral

$$a(\xi; \lambda) = \int_0^\xi e^{-\lambda\alpha}\, da(\alpha)$$

exists for finite complex values of λ and finite positive values of ξ. If, for a particular λ, $\lim_{\xi\to\infty} a(\xi; \lambda)$ exists, we denote the limit by

$$f(\lambda) = \int_0^\infty e^{-\lambda\xi}\, da(\xi). \tag{10.2.1}$$

We say that the integral *converges* for this value of λ and call $f(\lambda)$ the *Laplace-Stieltjes transform* of $a(\xi)$. If $a(\xi)$ is absolutely continuous and $a(\xi) = \int_0^\xi g(\alpha)d\alpha$, then

$$f(\lambda) = \int_0^\infty e^{-\lambda\xi} g(\xi)\, d\xi \tag{10.2.2}$$

and $f(\lambda)$ is called the *Laplace transform* of $g(\xi)$.

Let $a_*(\xi)$ denote the *strong variation* of $a(\alpha)$ in $[0, \xi]$. We say that the Laplace-Stieltjes integral *is absolutely convergent* for $\lambda = \lambda_0$ if the numerical integral

$$\phi(\lambda) = \int_0^\infty e^{-\lambda\xi}\, da_*(\xi) \tag{10.2.3}$$

converges for $\lambda = \lambda_0$. This implies ordinary convergence and the inequality

$$\| f(\sigma + i\tau) \| \leqq \phi(\sigma) \tag{10.2.4}$$

whenever the right side exists. Necessary and sufficient conditions for the convergence of (10.2.3) may of course be read off from the classical theory and the latter also indicates the nature of the results relating to ordinary convergence in the abstract case.

The following approach to the convergence theory was developed by F. Bohnenblust in 1932 (unpublished) for the case of numerically-valued functions. Let $a(\infty)$ denote $\lim_{\xi\to\infty} a(\xi)$, whenever the latter exists as an element of $\mathfrak{X}$, otherwise θ. We define the *order* of $a(\xi)$ to be

$$\gamma = \limsup_{\xi\to\infty} \frac{1}{\xi} \log \| a(\xi) - a(\infty) \|. \tag{10.2.5}$$

The functions $a(\xi)$ of finite order form a linear space $\mathfrak{A}$; we collect functions having the same order γ into a subspace $\mathfrak{A}(\gamma)$. Since the order of the sum of two functions of order γ is $\leqq \gamma$, the spaces $\mathfrak{A}(\gamma)$ are not linear.

We observe next that $a(\xi; \lambda)$ is the strong limit of properly chosen Riemann sums; hence integration by parts is permissible and gives

$$a(\xi; \lambda) = e^{-\lambda\xi} a(\xi) + \lambda \int_0^\xi e^{-\lambda\alpha} a(\alpha)\, d\alpha, \tag{10.2.6}$$

where the integral now exists in the Riemann-Graves sense. Conversely we have

$$\begin{aligned} a(\xi) &= a(\xi; 0) = \int_0^\xi e^{\lambda\alpha} d_\alpha a(\alpha; \lambda) \\ &= e^{\lambda\xi} a(\xi; \lambda) - \lambda \int_0^\xi e^{\lambda\alpha} a(\alpha; \lambda)\, d\alpha. \end{aligned} \tag{10.2.7}$$

Formula (10.2.6) defines a linear transformation

$$T(\lambda)[a(\xi)] = a(\xi; \lambda)$$

on $\mathfrak{A}$ to itself which maps the subspace $\mathfrak{A}(\gamma)$ on the subspace $\mathfrak{A}[\gamma - \mathfrak{R}(\lambda)]$ in a one-to-one manner. Indeed, if $a(\xi) \in \mathfrak{A}(\gamma)$, then (10.2.6) shows that the order of $T(\lambda)[a(\xi)]$ is $\leqq \gamma - \mathfrak{R}(\lambda)$ and (10.2.7), which defines the inverse transformation, shows that inequality is excluded and that the correspondence between the two subspaces is one-to-one. A simple computation shows that $T(\lambda)$ has the group property

$$T(\lambda_1 + \lambda_2) = T(\lambda_1)T(\lambda_2). \tag{10.2.8}$$

The functions $a(\xi)$ for which $\lim_{\xi\to\infty} a(\xi)$ exists form a certain linear subspace $\mathfrak{C}$ of $\mathfrak{A}$. A sufficient condition that $a(\xi) \in \mathfrak{C}$ is that the order of $a(\xi)$ is < 0 and a necessary condition is that it is $\leqq 0$. Now *the Laplace-Stieltjes transform of $a(\xi)$ will exist for a particular λ if and only if $T(\lambda)[a(\xi)] \in \mathfrak{C}$*, that is, *if* the order of $T(\lambda)[a(\xi)]$ is < 0 and *only if* it is $\leqq 0$. From this observation we get

THEOREM 10.2.1. *There exist two real numbers σ_0 and σ_a such that the integral (10.2.1) is convergent for $\mathfrak{R}(\lambda) > \sigma_0$, but not for any λ with $\mathfrak{R}(\lambda) < \sigma_0$, and it is absolutely convergent for $\mathfrak{R}(\lambda) > \sigma_a$, but not for any λ with $\mathfrak{R}(\lambda) < \sigma_a$. We have*

$$-\infty \leqq \sigma_0 \leqq \sigma_a \leqq \infty, \tag{10.2.9}$$

and

$$\sigma_0 = \limsup_{\xi\to\infty} \frac{1}{\xi} \log \| a(\infty) - a(\xi) \|, \tag{10.2.10}$$

$$\sigma_a = \limsup_{\xi\to\infty} \frac{1}{\xi} \log | a_*(\infty) - a_*(\xi) |, \tag{10.2.11}$$

where $a(\infty) = \lim_{\xi\to\infty} a(\xi)$ or θ according as the limit exists or not and $a_*(\infty)$ is defined similarly.

We refer to σ_0 and σ_a as the *abscissas of ordinary* and of *absolute convergence* respectively. A different abscissa of absolute convergence could be defined by replacing the strong variation by the total variation in (10.2.11). There is also an *abscissa of uniform convergence* which lies between σ_0 and σ_a. We do not wish to introduce these notions here, especially as they may lead to added confusion in the case in which $a(\xi)$ is an operator where we have to consider the uniform as well as the strong topology.

Formula (10.2.6) shows that for $\Re(\lambda) > \sigma_0$

$$f(\lambda) = \lambda \int_0^\infty e^{-\lambda\xi}[a(\xi) - a(\infty)]\, d\xi + a(\infty). \tag{10.2.12}$$

If $\Re(\lambda) > \max(\sigma_0, 0)$ we may omit $a(\infty)$ in this formula. We conclude that $f(\lambda)$ is holomorphic for $\Re(\lambda) > \sigma_0$. Further

$$\| f(\lambda) \| = O(|\lambda|), \qquad \Re(\lambda) > \sigma_0 + \epsilon.$$

From (10.2.4) we get that $f(\lambda)$ is uniformly bounded when $\Re(\lambda) > \sigma_a + \epsilon$. The derivatives of $f(\lambda)$ are given by

$$f^{(n)}(\lambda) = (-1)^n \int_0^\infty e^{-\lambda\xi}\xi^n\, da(\xi), \qquad \Re(\lambda) > \sigma_0. \tag{10.2.13}$$

Since $f(\lambda)$ is holomorphic in a half-plane and its norm cannot grow faster than $O(|\lambda|)$, it follows from the extensions of the principle of the maximum (see section 3.12) that its zeros must be distributed rather sparingly if the function is not to vanish identically. The classical instance of this observation is the *theorem of Lerch* which holds also for abstract Laplace-Stieltjes integrals:

THEOREM 10.2.2. *If $f(\lambda)$ is holomorphic in $\Re(\lambda) > \sigma_0$ and if $f(\lambda) = \theta$ for $\lambda = \lambda_0 + n, n = 1, 2, 3, \cdots$, then $f(\lambda) \equiv \theta$.*

PROOF. If $x^* \in \mathfrak{X}^*$ is a linear bounded functional, then

$$x^*[f(\lambda)] = \lambda \int_0^\infty e^{-\lambda\xi} x^*[a(\xi)]\, d\xi, \qquad \Re(\lambda) > \max(0, \sigma_0),$$

vanishes for $\lambda = \lambda_0 + n$ and, by the classical theorem of Lerch, this implies $x^*[f(\lambda)] \equiv 0$. Since this holds for every x^* we have $f(\lambda) \equiv \theta$.

We note that $x^*[a(\xi)]$ is also normalized. Hence by the classical uniqueness theorem for Laplace-Stieltjes integrals $x^*[a(\xi)] \equiv 0$ (not merely for almost all ξ) and this implies $a(\xi) \equiv \theta$. Thus the *uniqueness theorem* also extends:

THEOREM 10.2.3. *There cannot exist two different normalized representations of $f(\lambda)$ in terms of Laplace-Stieltjes integrals.*

In the classical theory it is shown that the product of two absolutely convergent Laplace-Stieltjes integrals is an integral of the same kind. Since only scalar multiplication has a sense in ordinary (B)-spaces, we have to be satisfied with a partial extension of this theorem. To simplify matters we introduce a slight restriction in the scalar factor.

THEOREM 10.2.4. *Let*

$$\gamma(\lambda) = \int_0^\infty e^{-\lambda\xi}\, d\beta(\xi), \qquad f(\lambda) = \int_0^\infty e^{-\lambda\xi}\, da(\xi)$$

be Laplace-Stieltjes integrals absolutely convergent for $\Re(\lambda) > \sigma_a$. *Here* $\beta(\xi)$ *is to be a continuous numerically-valued function of bounded variation,* $\beta(0) = 0$, *and* $a(\xi)$ *is a normalized function of strongly bounded variation on* $[0, \infty)$ *to* $\mathfrak{X}$. *Then*

$$\gamma(\lambda)f(\lambda) = \int_0^\infty e^{-\lambda\xi}\, dc(\xi) \tag{10.2.14}$$

with

$$c(\xi) = \int_0^\xi a(\xi - \eta)\, d\beta(\eta) = \int_0^\xi \beta(\xi - \eta)\, da(\eta), \tag{10.2.15}$$

the integral being absolutely convergent for $\Re(\lambda) > \sigma_a$.

PROOF. We note that $c(\xi)$ is defined for all ξ since $\beta(\xi)$ is continuous, the two expressions for $c(\xi)$ being found to be equal by an integration by parts. We define $\beta(\xi) \equiv 0$ and $a(\xi) \equiv \theta$ for $\xi < 0$. We denote the variation of $\beta(\xi)$ in $(-\infty, \eta)$ by $\beta_*(\eta)$ with similar notation for the strong variations of $a(\xi)$ and $c(\xi)$. We have then

$$c(\xi) = \int_{-\infty}^\infty \beta(\xi - \eta)\, da(\eta)$$

and from this representation one concludes successively that (i) $c(0) = \theta$, (ii) $c(\xi)$ is continuous, (iii) $c(\xi)$ is of strongly bounded variation, and (iv) $c_*(\xi) \leqq \beta_*(\xi)a_*(\xi)$ for all ξ. Without restricting the generality we may assume $\sigma_a = 0$. We have then $\log \beta_*(\omega) = o(\omega)$, $\log a_*(\omega) = o(\omega)$ so, by (iv), $\log c_*(\omega) = o(\omega)$ whence it follows that (10.2.14) is absolutely convergent for $\Re(\lambda) > 0 = \sigma_a$. To prove that the integral really represents the product $\gamma(\lambda)f(\lambda)$ it is enough to show that the linear functionals of the two sides of (10.2.14) are equal and this follows from the classical multiplication theorem for Laplace-Stieltjes integrals. This completes the proof.

The particular case $\gamma(\lambda) = \lambda^{-\alpha}$ leads to the important representation

$$f(\lambda) = \lambda^\alpha \int_0^\infty e^{-\lambda\xi}\, d_\xi\, a_\alpha(\xi), \qquad \Re(\lambda) > \max(0, \sigma_0), \tag{10.2.16}$$

where

$$a_\alpha(\xi) = \frac{1}{\Gamma(\alpha)} \int_0^\xi (\xi - \eta)^{\alpha - 1} a(\eta)\, d\eta \tag{10.2.17}$$

is the fractional integral of $a(\xi)$ of order α, $\Re(\alpha) > 0$. This is a generalization of (10.2.12) using integration by parts of fractional order. The main value of this formula lies in the fact that it defines a fairly effective method of summation

for divergent Laplace-Stieltjes integrals which is equivalent to the arithmetic means (C, α) in the right half-plane. The effectiveness is based upon the fact that if $f(\lambda)$ is of finite order and holomorphic in a half-plane, then $\lambda^{-\alpha}f(\lambda)$ is representable by an absolutely convergent Laplace-Stieltjes integral for sufficiently large values of $\Re(\alpha)$ (cf. section 10.6).

It is important to realize that (10.2.16) is actually a convergent representation of $f(\lambda)$ for $\Re(\lambda) > \max(0, \sigma_0)$; this does not follow from Theorem 10.2.4 which asserts something different, namely absolute convergence for $\Re(\lambda) > \max(0, \sigma_a)$. However, it is an easy matter to show that the integral converges for $\Re(\lambda) > \max(0, \sigma_0)$ by estimating the rate of growth of $a_\alpha(\xi)$, that of $a(\xi)$ being known. To prove that the integral equals $\lambda^{-\alpha}f(\lambda)$ it is enough to show that the linear functionals are equal and this follows from another classical multiplication theorem for Laplace-Stieltjes integrals according to which the product integral converges if one of the factors converges absolutely, the other being merely convergent. In our case $\lambda^{-\alpha}$ has an absolutely convergent representation for $\Re(\lambda) > 0$.

10.3. Inversion formulas. For the complex inversion formulas we need some preliminaries. With a slight change of conventional notation we place

$$\text{si}(\xi) = \frac{1}{\pi}\int_\xi^\infty \frac{\sin\eta}{\eta}\,d\eta. \tag{10.3.1}$$

The following properties of $\text{si}(\xi)$ are well known. It is a bounded function whose maximum $M = 1.0894\cdots$ is reached for $\xi = -\pi$, while the minimum $-0.28\cdots$ is reached at $\xi = \pi$. For small values of ξ

$$\text{si}(\xi) = \tfrac{1}{2} + O(\xi),$$

for large values

$$\text{si}(\xi) = O\left(\frac{1}{\xi}\right), \qquad \xi \to +\infty, \qquad \text{si}(\xi) = 1 + O\left(\frac{1}{\xi}\right), \qquad \xi \to -\infty.$$

We shall now prove the following lemma for Dirichlet's integral which is well known in the numerical case:

Lemma 10.3.1. *Let $\mathfrak{B}$ be the class of functions $b(\xi)$ on $(-\infty, \infty)$ to $\mathfrak{X}$ such that $b(\xi) \to \theta$ when $\xi \to -\infty$ and $b(\xi)$ is of strongly bounded variation on $(-\infty, \infty)$. Then*

$$b(\xi \mid \omega) = \lim_{\alpha\to\infty} \frac{1}{\pi}\int_{-\alpha}^{\alpha} \frac{\sin\omega\eta}{\eta}\, b(\xi + \eta)\,d\eta \tag{10.3.2}$$

exists for all real ξ and ω and is absolutely continuous in ξ on every finite interval. Further

$$b(\xi \mid \omega) = \int_{-\infty}^{\infty} \text{si}(\omega\eta)\,d_\eta b(\xi + \eta) = \int_{-\infty}^{\infty} \text{si}[\omega(\sigma - \xi)]\,db(\sigma), \tag{10.3.3}$$

$$\| b(\xi \mid \omega) \| \leqq Mb_*(\infty), \tag{10.3.4}$$

$$\lim_{\omega\to\infty} b(\xi \mid \omega) = \tfrac{1}{2}[b(\xi - 0) + b(\xi + 0)] \tag{10.3.5}$$

for every ξ. The limit exists uniformly with respect to ξ in any closed interval of continuity of $b(\xi)$.

Remark. It is perhaps necessary to observe that $b(\xi \mid \omega)$ is ordinarily not a member of $\mathfrak{B}$. A counter example is given by the characteristic function of the interval (β, ∞) in the numerical case with $b(\xi \mid \omega) = \mathrm{si}\,[\omega(\beta - \xi)]$. It is true, however, that $\lim_{\xi \to -\infty} b(\xi \mid \omega) = \theta$ for all elements of $\mathfrak{B}$.

Proof. Formula (10.3.3) is proved by integration by parts under the limit sign in (10.3.2), and (10.3.4) is an immediate consequence thereof. We recall that $b_*(\sigma)$ is the strong variation of $b(\xi)$ in the interval $(-\infty, \sigma)$. That $b(\xi \mid \omega)$ is absolutely continuous in ξ follows by differentiation with respect to ξ under the sign of integration in the last member of (10.3.3) which leads to an absolutely convergent integral. To prove (10.3.5) we break up the interval of integration in the second member of (10.3.3) into six parts denoting the integrals from left to right by I_1 to I_6. The partition points are taken at $\xi = -\omega^{-\frac{1}{2}}, -\omega^{-2}, 0, \omega^{-2}, \omega^{-\frac{1}{2}}$. Using the properties of si (ξ) stated above, we see that

$$\begin{aligned} I_1 &= b(\xi - \omega^{-\frac{1}{2}}) + O(\omega^{-\frac{1}{2}}), \\ \| I_2 \| &\leqq M[b_*(\xi - \omega^{-2}) - b_*(\xi - \omega^{-\frac{1}{2}})], \\ I_3 &= \tfrac{1}{2}[b(\xi) - b(\xi - \omega^{-2})] + O(\omega^{-1}), \\ I_4 &= \tfrac{1}{2}[b(\xi + \omega^{-2}) - b(\xi)] + O(\omega^{-1}), \\ \| I_5 \| &\leqq M[b_*(\xi + \omega^{-\frac{1}{2}}) - b_*(\xi + \omega^{-2})], \\ I_6 &= O(\omega^{-\frac{1}{2}}), \end{aligned}$$

where all the O-estimates hold uniformly with respect to ξ in $(-\infty, \infty)$. From these relations (10.3.5) follows. This completes the proof.

We can now prove the *complex inversion formula:*

Theorem 10.3.1. *Let $f(\lambda)$ be defined by (10.2.1), convergent for $\Re(\lambda) > \sigma_0$, let $\gamma > \max(0, \sigma_0)$ and set*

$$a(\xi \mid \omega) = \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda\xi} f(\lambda) \frac{d\lambda}{\lambda}. \tag{10.3.6}$$

Then

$$\lim_{\omega \to \infty} a(\xi \mid \omega) = \begin{cases} a(\xi), & \xi > 0, \\ \tfrac{1}{2}\, a(+0), & \xi = 0, \\ 0, & \xi < 0. \end{cases} \tag{10.3.7}$$

The limit exists uniformly with respect to ξ in any finite interval of continuity of $a(\xi)$.

Proof. We substitute (10.2.12) for $f(\lambda)$ in the definition of $a(\xi \mid \omega)$ and interchange the order of integration as we may do using the Fubini theorem. It follows that

(10.3.8) $$a(\xi \mid \omega) = \frac{1}{\pi} \int_0^\infty a(\eta)\, e^{\gamma(\xi-\eta)} \frac{\sin \omega(\xi - \eta)}{\xi - \eta}\, d\eta.$$

Since $\gamma > \sigma_0$, formula (10.2.10) shows that the integral is absolutely convergent. Setting $b(\xi) = a(\xi)e^{-\gamma\xi}$ for $\xi \geqq 0$ and $= \theta$ for $\xi < 0$ with $b(\xi \mid \omega) = a(\xi \mid \omega)e^{-\gamma\xi}$, we see that $b(\xi)$ satisfies the conditions of the preceding lemma and that $b(\xi \mid \omega)$ is defined by formula (10.3.2). Since $a(\xi)$ is normalized for $\xi \neq 0$, so is $b(\xi)$, and the desired result follows from the lemma.

Applying this theorem to formula (10.2.16), we see that for $\Re(\alpha) > 0, \xi > 0$,

(10.3.9) $$\lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda\xi} f(\lambda) \frac{d\lambda}{\lambda^{1+\alpha}} = \frac{1}{\Gamma(\alpha)} \int_0^\xi (\xi - \eta)^{\alpha-1} a(\eta)\, d\eta,$$

the limit being θ for $\xi < 0$.

For the applications which we have in mind an inversion formula for Laplace integrals will also be needed.

Theorem 10.3.2. *Let $g(\xi) \in B[(0, \omega); \mathfrak{X}]$ for every finite ω and let*

(10.3.10) $$f(\lambda) = \int_0^\infty e^{-\lambda\xi} g(\xi)\, d\xi$$

be absolutely convergent for $\Re(\lambda) > \sigma_a$. Let $\gamma > \max(0, \sigma_a)$ and set

(10.3.11) $$g_1(\xi \mid \omega) = \frac{1}{2\pi} \int_{-\omega}^{\omega} \left\{1 - \frac{|\tau|}{\omega}\right\} e^{(\gamma+i\tau)\xi} f(\gamma + i\tau)\, d\tau.$$

Then $\lim_{\omega\to\infty} g_1(\xi \mid \omega) = g(\xi)$ for almost all positive ξ and equals θ for $\xi < 0$. The limit equals $\frac{1}{2}[g(\xi + 0) + g(\xi - 0)]$ whenever this expression has a meaning. The limit exists uniformly with respect to ξ in any finite interval of continuity of $g(\xi)$.

Proof. We substitute (10.3.10) for $f(\lambda)$ in the definition of $g_1(\xi \mid \omega)$ and interchange the order of integration obtaining after some simplification

$$g_1(\xi \mid \omega) = \frac{2}{\pi\omega} \int_0^\infty \frac{\sin^2 [\omega(\xi - \alpha)/2]}{(\xi - \alpha)^2} g(\alpha) e^{\gamma(\xi-\alpha)}\, d\alpha.$$

Put $h(\xi) = g(\xi)e^{-\gamma\xi}$ or θ according as $\xi > 0$ or < 0 with $h_1(\xi \mid \omega) = g_1(\xi \mid \omega)e^{-\gamma\xi}$. Since $\gamma > \sigma_a$, $h(\xi) \in B[E_1; \mathfrak{X}]$ and

$$h_1(\xi \mid \omega) = \int_0^\infty F(\xi - \alpha; \omega) h(\alpha)\, d\alpha$$

with

$$F(\beta; \omega) = \frac{2}{\pi\omega} \frac{\sin^2 (\omega\beta/2)}{\beta^2}.$$

It is a simple matter to verify that the Fejér kernel satisfies all the conditions of Theorems 3.7.1 to 3.7.3 and condition (4′) with $\mu_1 = \mu_2 = \frac{1}{2}$. It follows that

$h_1(\xi \mid \omega)$ tends to $h(\xi)$ when $\omega \to \infty$, in the mean of order one as well as pointwise in the Lebesgue set of $h(\xi)$. Since the Lebesgue sets of $g(\xi)$ and $h(\xi)$ are identical, the conclusions of the theorem are immediate.

The theorem shows that the integral

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda\xi} f(\lambda)\, d\lambda$$

is summable $(C, 1)$ to $g(\xi)$ almost everywhere if $f(\lambda)$ is the Laplace transform of $g(\xi)$ and $\gamma > \max(0, \sigma_a)$. It is clear that other methods of summation could be used for the same purpose.

The two preceding theorems express the *determining function* $a(\xi)$ or $g(\xi)$ in terms of the values of the *generating function* $f(\lambda)$ on vertical lines in the half-plane of convergence. There are many other means at our disposal for the solution of the inversion problem. Methods involving the values of $f(\lambda)$ at the positive integers or the values of $f(\lambda)$ and its derivatives at some fixed point are available but will not be considered here. Another class of methods involves the values of $f(\lambda)$ for large real values of λ. Here the oldest method is that of E. Phragmén [1] which extends easily to the abstract case.

THEOREM 10.3.3. *Let $f(\lambda)$ satisfy the assumptions of Theorem* 10.3.1 *and set*

$$a_0(\xi \mid \omega) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} f(n\omega) e^{n\omega\xi}. \tag{10.3.12}$$

Then

$$\lim_{\omega \to \infty} a_0(\xi \mid \omega) = \begin{cases} (1 - e^{-1}) a(\xi - 0) + e^{-1} a(\xi + 0), & \xi > 0, \\ e^{-1} a(+0), & \xi = 0, \\ \theta, & \xi < 0. \end{cases}$$

If $a(\xi)$ is absolutely continuous, $a'(\xi) = g(\xi)$, then

$$\lim_{\omega \to \infty} \frac{d}{d\xi} a_0(\xi \mid \omega) = g(\xi), \quad \xi > 0,$$

in the Lebesgue set of $g(\xi)$.

PROOF. Here we have the representation

$$a_0(\xi \mid \omega) = \int_0^{\infty} E(\xi - \alpha; \omega) a(\alpha)\, d\alpha, \tag{10.3.13}$$

where

$$E(\beta; \omega) = \omega \exp\,[\omega\beta - e^{\omega\beta}]$$

is the derivative with respect to β of the discontinuous factor of H. von Koch. It is an easy matter to verify that this kernel satisfies the conditions of Theorem 3.7.2 including condition (4′) with $\mu_1 = 1 - e^{-1}$, $\mu_2 = e^{-1}$. This proves the first

assertion. Replacing $a(\xi)$ by $g(\xi)$ in (10.3.13), the value of the integral becomes $da_0(\xi \mid \omega)/d\xi$ instead. But $E(\beta;\omega)$ satisfies the conditions of Theorem 3.7.3, taking $P(\beta;\omega) = \omega e^{-\omega\beta}$, and this proves the second assertion.

A different class of inversion formulas has been investigated very thoroughly by D. V. Widder after preliminary work by E. L. Post. We shall consider only one of Widder's operators

$$L_{k,\xi}[f(\lambda)] = \frac{(-1)^k}{k!} f^{(k)}\left(\frac{k}{\xi}\right)\left(\frac{k}{\xi}\right)^{k+1} \tag{10.3.14}$$

which is defined for large values of k for all $\xi > 0$.

Theorem 10.3.4. *If $f(\lambda)$ is the Laplace transform of $g(\xi)$*

$$\lim_{k\to\infty} L_{k,\xi}[f(\lambda)] = g(\xi)$$

in the Lebesgue set of $g(\xi)$. If $f(\lambda)$ is the Laplace-Stieltjes transform of $a(\xi)$ then

$$\lim_{k\to\infty} \int_0^\xi L_{k,\tau}[f(\lambda)]\,d\tau = a(\xi) - a(+0).$$

Proof. In the first case we are led to the representation

$$L_{k,\xi}[f(\lambda)] = \frac{1}{\xi}\int_0^\infty W_0\left(\frac{\sigma}{\xi};k\right) g(\sigma)\,d\sigma,$$

where

$$W_0(\beta;k) = \frac{k^{k+1}}{k!}\beta^k e^{-k\beta}.$$

The substitutions $\sigma = e^\tau$, $\xi = e^\eta$, $\beta = e^\gamma$ lead to the singular integral

$$\int_{-\infty}^\infty W(\tau - \eta;k) g(e^\tau)\,d\tau,$$

where

$$W(\gamma;k) = \frac{k^{k+1}}{k!}\exp\,[(k+1)\gamma - ke^\gamma].$$

This kernel satisfies the conditions of Theorem 3.7.3 if we take $P(\gamma;k) = k^{\frac{1}{2}}\exp\,[-k\gamma^2/2]$ so that the limit of the integral is $g(e^\eta) = g(\xi)$ in the Lebesgue set. Since the conditions of Theorem 3.7.2 also hold, including (4′) with $\mu_1 = \mu_2 = \frac{1}{2}$, we have

$$\lim_{k\to\infty} L_{k,\xi}[f(\lambda)] = \tfrac{1}{2}[g(\xi+0) + g(\xi-0)]$$

whenever this expression has a meaning.

For the proof of the second part we note that

$$L_{k,\xi}[f(\lambda)] = \frac{1}{\xi}\int_0^\infty W_0\left(\frac{\sigma}{\xi}; k\right) da(\sigma) = \int_{-\infty}^\infty e^{-\tau} W(\tau - \eta; k)\, da(e^\tau)$$

$$= -\int_{-\infty}^\infty \frac{\partial}{\partial\tau}[e^{-\tau}W(\tau - \eta; k)]a(e^\tau)\, d\tau$$

$$= \int_{-\infty}^\infty e^{-\tau-\eta}\frac{\partial}{\partial\eta}[e^\eta W(\tau - \eta; k)]a(e^\tau)\, d\tau.$$

Hence

$$\int_{\xi_1}^{\xi_2} L_{k,\xi}[f(\lambda)]\, d\xi = \int_{\eta_1}^{\eta_2} L_{k,\xi}[f(\lambda)]e^\eta\, d\eta$$

$$= \int_{\eta_1}^{\eta_2}\int_{-\infty}^\infty e^{-\tau}\frac{\partial}{\partial\eta}[e^\eta W(\tau - \eta; k)]a(e^\tau)\, d\tau\, d\eta$$

$$= \int_{-\infty}^\infty e^{-\tau}[e^\eta W(\tau - \eta; k)]_{\eta_1}^{\eta_2} a(e^\tau)\, d\tau.$$

When $k \to \infty$ this expression tends to

$$a(e^{\eta_2}) - a(e^{\eta_1}) = a(\xi_2) - a(\xi_1)$$

by virtue of the Remark appended to Theorem 3.7.2. Here we have assumed that $a(\xi)$ is normalized as usual. For $\xi_2 = \xi$, $\xi_1 \to 0$, we obtain the limit $a(\xi) - a(+0)$ which is the required result. The interchange of the limit passages $k \to \infty$ and $\xi_1 \to 0$ can be justified and it can also be shown that $L_{k,\xi}[f(\lambda)]$ is absolutely integrable down to zero. We omit these details. An important consequence of the theorem is the following

Theorem 10.3.5. *If $f(\lambda)$ is the Laplace transform of $g(\xi)$, then $\|g(\xi)\| \leqq M$ for almost all $\xi > 0$ if and only if for all $\lambda > 0$ and $k = 0, 1, 2, \cdots$ we have*

$$\lambda^{k+1}\|f^{(k)}(\lambda)\| \leqq Mk!. \tag{10.3.15}$$

If $f(\lambda)$ is the Laplace-Stieltjes transform of $a(\xi)$ then $a_*(\infty) \leqq M + \|f(\infty)\|$ if and only if for $k = 1, 2, 3, \cdots$

$$\int_0^\infty \lambda^{k-1}\|f^{(k)}(\lambda)\|\, d\lambda \leqq M(k-1)!. \tag{10.3.16}$$

Proof. In the first case

$$\lambda^{k+1}\|f^{(k)}(\lambda)\| = \left\| \int_0^\infty e^{-\lambda\xi}(\lambda\xi)^k g(\xi)\, d(\lambda\xi)\right\| \leqq Mk!$$

if $\|g(\xi)\| \leqq M$ almost everywhere, so (10.3.15) is necessary. In the second case

$$\int_0^\infty \lambda^{k-1}\|f^{(k)}(\lambda)\|\, d\lambda \leqq \int_0^\infty\int_0^\infty e^{-\lambda\xi}(\lambda\xi)^{k-1}\xi\, d\lambda\, da_*(\xi) = (k-1)!\int_{0+}^\infty da_*(\xi)$$

$$= (k-1)![a_*(\infty) - a_*(+0)] \leqq M(k-1)!,$$

where we have used that $a_*(+0) = \|f(\infty)\|$. Thus (10.3.16) is necessary. To prove the sufficiency we note that the first condition implies $\| L_{k,\xi}[f(\lambda)] \| \leqq M$ for all ξ, the second that $\int_0^\infty \| L_{k,\tau}[f(\lambda)] \| \, d\tau \leqq M$; since $f(\lambda)$ is assumed to be the Laplace transform of $g(\xi)$ in the first case and a Laplace-Stieltjes transform in the second, the preceding theorem ensures that $\| g(\xi) \| \leqq M$ for almost all ξ or $a_*(\infty) - a_*(+0) \leqq M$ respectively.

Similar conditions can be formulated for the case in which $g(\xi) \in B_p[(0, \infty); \mathfrak{X}]$, $1 \leqq p < \infty$, and $f(\lambda)$ is the Laplace transform of $g(\xi)$. Widder has shown in the numerical case that the conditions of Theorem 10.3.5 and its extensions are sufficient to ensure that the function $f(\lambda)$ be a Laplace or a Laplace-Stieltjes transform. His argument seems to hold for any abstract space in which bounded sets are weakly compact, but it is not clear at the time of the present writing whether or not his conditions are always sufficient for the existence of the representation.

2. Functions Holomorphic in a Half-Plane

10.4. The classes $H_p(\alpha; \mathfrak{X})$. We shall investigate various classes of functions $f(\lambda)$ which are holomorphic in a fixed half-plane $\sigma > \alpha$, $\lambda = \sigma + i\tau$, and have values in a fixed complex (B)-space $\mathfrak{X}$. In addition $f(\lambda)$ will be subjected to different types of boundedness conditions. We start with the classes $H_p(\alpha; \mathfrak{X})$; for the properties of numerically-valued functions of the class $H_p(\alpha)$ which will be used in the following, we refer to the papers by E. Hille and J. D. Tamarkin where further literature is quoted.

Definition 10.4.1. $f(\lambda) \in H_p(\alpha; \mathfrak{X})$, *p fixed*, $1 \leqq p < \infty$, *if*

(i) $f(\lambda)$ *is a function on complex numbers to* $\mathfrak{X}$ *which is holomorphic for* $\sigma > \alpha$;

(ii) $\sup_{\sigma > \alpha} \{\int_{-\infty}^{\infty} \| f(\sigma + i\tau) \|^p \, d\tau\}^{1/p} \equiv \| f \|_p < \infty$;

(iii) $\lim_{\sigma \to \alpha} f(\sigma + i\tau) \equiv f(\alpha + i\tau)$ *exists for almost all values of* τ *and* $f(\alpha + i\tau) \in B_p[(-\infty, \infty); \mathfrak{X}]$.

For the definition of the class $B_p[S; \mathfrak{X}]$, $S = (-\infty, \infty)$, see the remarks after Theorem 3.6.4. Assumption (iii) is probably redundant; we assume it explicitly to obviate the need of a lengthy digression. It is known that $f(\sigma + i\tau)$ converges weakly when $\sigma \to \alpha$ for almost all τ. More precisely, for every linear bounded functional $x^* \in \mathfrak{X}^*$, we have $x^*[f(\lambda)] \in H_p(\alpha)$ and $\lim_{\sigma \to \alpha} x^*[f(\sigma + i\tau)]$ exists for almost all τ, where, however, the exceptional set may depend upon x^*, and the limit function belongs to $L_p(-\infty, \infty)$.

Theorem 10.4.1. *If* $f(\lambda) \in H_p(\alpha; \mathfrak{X})$, *then* $f(\lambda)$ *is represented by its proper Cauchy and Poisson integrals for* $\sigma > \alpha$, *that is*

$$(10.4.1) \qquad f(\lambda) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{f(\mu)}{\lambda-\mu}\,d\mu,$$

$$(10.4.2) \qquad f(\sigma + i\tau) = \frac{\sigma-\alpha}{\pi}\int_{-\infty}^{\infty} \frac{f(\alpha+i\beta)\,d\beta}{(\sigma-\alpha)^2+(\tau-\beta)^2}.$$

PROOF. The integrals exist by virtue of (iii). In order to prove that they have the value $f(\lambda)$, it is enough to observe that the functionals agree. Thus

$$x^*\left\{\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{f(\mu)}{\lambda-\mu}\,d\mu\right\} = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{x^*[f(\mu)]}{\lambda-\mu}\,d\mu = x^*[f(\lambda)]$$

since $x^*[f(\lambda)] \in H_p(\alpha)$. This being true for every x^*, formula (10.4.1) must hold and similarly for (10.4.2).

THEOREM 10.4.2. *If $f(\lambda) \in H_p(\alpha; \mathfrak{X})$, then*

$$(10.4.3) \qquad \|f(\sigma+i\tau)\| \leqq [\pi(\sigma-\alpha)]^{-1/p}\,\|f\|_p,$$

and for fixed $\delta > 0$

$$(10.4.4) \qquad \lim_{\rho\to\infty}\|f(\alpha+\delta+\rho e^{i\psi})\| = 0, \qquad -\frac{\pi}{2} \leqq \psi \leqq \frac{\pi}{2},$$

uniformly in ψ.

PROOF. The inequality follows from (10.4.2). The integral of the Poisson kernel over the range $-\infty < \tau < \infty$ being identically one, a classical use of Hölder's inequality gives

$$(10.4.5) \qquad \begin{aligned} \|f(\sigma+i\tau)\|^p &\leqq \frac{\sigma-\alpha}{\pi}\int_{-\infty}^{\infty}\frac{\|f(\alpha+i\beta)\|^p\,d\beta}{(\sigma-\alpha)^2+(\tau-\beta)^2} \\ &\leqq [\pi(\sigma-\alpha)]^{-1}\int_{-\infty}^{\infty}\|f(\alpha+i\beta)\|^p\,d\beta \\ &= [\pi(\sigma-\alpha)]^{-1}[\|f\|_p]^p, \end{aligned}$$

which is the required estimate. From this it follows that (10.4.4) holds uniformly in every sector $-\frac{1}{2}\pi + \epsilon \leqq \psi \leqq \frac{1}{2}\pi - \epsilon$. In order to prove the sharper assertion made above, we assume $\tau > 0$, and write the integral in the second member of (10.4.5) as the sum of two integrals, one from $-\infty$ to $\tau/2$ and the other from $\tau/2$ to ∞. Thus

$$\begin{aligned} \|f(\sigma+i\tau)\|^p &\leqq 4\frac{\sigma-\alpha}{\pi\tau^2}\int_{-\infty}^{\tau/2}\|f(\alpha+i\beta)\|^p\,d\beta + \frac{1}{\pi(\sigma-\alpha)}\int_{\tau/2}^{\infty}\|f(\alpha+i\beta)\|^p\,d\beta \\ &\leqq 4\frac{\sigma-\alpha}{\pi\tau^2}[\|f\|_p]^p + \frac{1}{\pi(\sigma-\alpha)}\int_{\tau/2}^{\infty}\|f(\alpha+i\beta)\|^p\,d\beta \end{aligned}$$

and this tends uniformly to zero when $\lambda \to \infty$ in such a manner that $\sigma \geqq \alpha + \delta$ and $\sigma\tau^{-2} \to 0$. There is a similar inequality for negative values of τ. These inequalities together with (10.4.3) suffice to prove (10.4.4).

THEOREM 10.4.3. *If $f(\lambda) \in H_p(\alpha; \mathfrak{X})$, then*
(i) $\lim_{\sigma\to\alpha} \int_{-\infty}^{\infty} \| f(\sigma + i\tau) - f(\alpha + i\tau) \|^p \, d\tau = 0$,
(ii) $T(\sigma; f) \equiv \int_{-\infty}^{\infty} \| f(\sigma + i\tau) \|^p \, d\tau$ *is a continuous monotone decreasing function of σ for $\sigma \geqq \alpha$. In particular,* $T(\alpha; f) = [\, \| f \|_p]^p$ *and* $T(\infty; f) = 0$.

PROOF. As in the classical case, one proves that the integral in (i) does not exceed

$$\frac{\sigma - \alpha}{\pi} \int_{-\infty}^{\infty} \frac{d\gamma}{(\sigma - \alpha)^2 + \gamma^2} \int_{-\infty}^{\infty} \| f(\alpha + i\beta) - f(\alpha + i\gamma + i\beta) \|^p \, d\beta.$$

By Theorems 3.6.3 and 6.10.1 the second integral is a continuous function of γ which equals zero when $\gamma = 0$. By a well-known property of the Poisson kernel, the repeated integral consequently tends to zero when $\sigma \to \alpha$ proving (i). The first two members of (10.4.5) give $T(\sigma; f) \leqq T(\alpha; f)$ for all $\sigma > \alpha$. But in the representation (10.4.2) we may replace α by any quantity σ_0, $\alpha < \sigma_0 < \sigma$, since every $f(\lambda)$ in $H_p(\alpha; \mathfrak{X})$ also belongs to $H_p(\sigma_0; \mathfrak{X})$. This gives $T(\sigma; f) \leqq T(\sigma_0; f)$ when $\sigma > \sigma_0$, so that $T(\sigma; f)$ is a decreasing function of σ. That $T(\sigma; f)$ is continuous for $\sigma = \alpha$ follows from (i). Since (i) also holds with α replaced by σ_0, $\alpha < \sigma_0 < \sigma$, we conclude that $T(\sigma; f) \to T(\sigma_0; f)$ when $\sigma \to \sigma_0$. This completes the proof since the statements concerning $T(\alpha; f)$ and $T(\infty; f)$ are obvious.

The class $H_p(\alpha; \mathfrak{X})$ is evidently linear; it becomes a metric space under the norm $\| f \|_p$ and it is a simple matter to prove that it is complete, so that it is a (B)-space.

10.5. Order relations. So far p was supposed to be finite. We say that $f(\lambda) \in H_\infty(\alpha; \mathfrak{X})$ *if it is holomorphic and bounded for $\sigma > \alpha$ and* $\lim_{\sigma\to\alpha} f(\sigma + i\tau)$ *exists for almost all τ. $H_\infty(\alpha; \mathfrak{X})$ is a (B)-space under the norm*

$$\| f \|_\infty = \sup_{\sigma > \alpha} \| f(\sigma + i\tau) \|.$$

The representation of $f(\lambda)$ by Poisson's integral, formula (10.4.2), is valid also when $p = \infty$ while most of the other results of section 10.4 become meaningless or false.

We shall need some properties of unbounded functions whose rates of growth are properly limited.

DEFINITION 10.5.1. *Let $f(\lambda)$ be a function on complex numbers to $\mathfrak{X}$, holomorphic for $\sigma \geqq \alpha$, except at infinity. Let $M(\rho; f) = \max \| f(\alpha + \rho e^{i\psi}) \|$ for $-\frac{1}{2}\pi \leqq \psi \leqq \frac{1}{2}\pi$. We say that $f(\lambda)$ is of finite order ω in this half-plane if*

$$\limsup_{\rho\to\infty} \log M(\rho; f)/\log \rho = \omega. \tag{10.5.1}$$

It is of finite order on vertical lines in the half-plane if

$$\limsup_{|\tau|\to\infty} \log \| f(\sigma + i\tau) \|/\log |\tau| \equiv \mu(\sigma) \tag{10.5.2}$$

is finite for $\sigma \geqq \alpha$. *Finally,* $f(\lambda)$ *is of exponential type in the half-plane if*

$$\limsup_{\rho\to\infty} \rho^{-1} \log M(\rho; f) = \beta < \infty. \tag{10.5.3}$$

It is of minimal type if $\beta = 0$, *normal type if* $\beta > 0$.

It is clear that if $f(\lambda)$ is of finite order in $\sigma \geqq \alpha$ then it is of finite order on vertical lines and $\mu(\sigma) \leqq \omega$. The following analog of a classical theorem of E. Lindelöf is less obvious.

THEOREM 10.5.1. *Let* $f(\lambda)$ *be holomorphic and of exponential minimal type in the half-plane* $\sigma \geqq \alpha$ *and suppose that* $\mu(\alpha)$ *is finite. Then* $f(\lambda)$ *is of finite order* $\mu(\alpha)$ *in the half-plane in question. Further,* $\mu(\sigma)$ *is monotonic non-increasing, convex, and continuous for* $\sigma \geqq \alpha$.

PROOF. Let γ be fixed, $\gamma > \mu(\alpha)$, and consider $g(\lambda) = \lambda^{-\gamma}f(\lambda)$. Without restricting the generality, we may assume that $\alpha > 0$ so that $g(\lambda)$ is holomorphic for $\sigma \geqq \alpha$ and bounded on $\sigma = \alpha$. By a classical Phragmén-Lindelöf argument or using Theorem 3.12.4 we see that $\| g(\lambda) \|$ is bounded for $\sigma \geqq \alpha$, that is, $f(\lambda)$ is of finite order $\omega \leqq \gamma$ in this half-plane. This being true for every $\gamma > \mu(\alpha)$, we have $\omega \leqq \mu(\alpha)$. On the other hand, $\| f(\alpha \pm i\rho) \| \leqq M(\rho; f)$, so that, comparing (10.5.1) with (10.5.2), we find that $\mu(\alpha) \leqq \omega$ and hence $\omega = \mu(\alpha)$.

By the preceding argument $\lambda^{-\gamma}f(\lambda)$ is bounded for $\gamma > \mu(\sigma_1)$ when $\sigma \geqq \sigma_1$ and, in particular, on the line $\sigma = \sigma_2$ if $\sigma_1 < \sigma_2$. From this we get that $\mu(\sigma_2) \leqq \mu(\sigma_1) + \epsilon$ for every $\epsilon > 0$. Hence $\mu(\sigma_2) \leqq \mu(\sigma_1)$ so that $\mu(\sigma)$ is never increasing. For the convexity proof we refer to Lindelöf's article [**2**, p. 3 et seq.]. A monotonic never increasing convex function being necessarily continuous, we have completed the proof.

Other growth measuring functions will be introduced in sections 10.7, 10.8, 13.3, and 13.6.

10.6. Representation by Laplace integrals. We shall show that every function on complex numbers to $\mathfrak{X}$ which is holomorphic and of finite order in a closed half-plane may be represented by a generalized Laplace integral. We start with functions of the class $H_p(\alpha; \mathfrak{X})$.

THEOREM 10.6.1. *Let* $f(\lambda) \in H_p(\alpha; \mathfrak{X})$ *where* $\alpha \geqq 0$. *Let* $\gamma > \alpha$ *and* $\beta p' > 1$ *where* $1/p + 1/p' = 1$. *Then*

$$a_\beta(\xi) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\mu\xi} \mu^{-\beta} f(\mu)\, d\mu \tag{10.6.1}$$

defines a continuous function on $(0, \infty)$ *to* $\mathfrak{X}$ *and*

$$f(\lambda) = \lambda^{\beta} \int_0^{\infty} e^{-\lambda\xi} a_\beta(\xi)\, d\xi, \tag{10.6.2}$$

the integral being absolutely convergent for $\sigma > \alpha$.

PROOF. That the integral defining $a_\beta(\xi)$ exists follows from Hölder's inequality. Since the integral converges uniformly with respect to ξ in every finite interval, $a_\beta(\xi)$ is continuous. The usual argument shows that the integral is independent of γ when $\gamma > \alpha$ and if $\alpha > 0$ we may even take $\gamma = \alpha$. In this case, a simple computation shows that $\| a_\beta(\xi) \| \leqq Me^{\alpha\xi}$; if $\alpha = 0$ we get instead $\| a_\beta(\xi) \| \leqq M(\epsilon)e^{\epsilon\xi}$ for every $\epsilon > 0$. It follows that the second integral is absolutely convergent for $\sigma > \alpha$. In order to find its value, we substitute the integral for $a_\beta(\xi)$, interchange the order of integration, and use (10.4.1). The details are left to the reader.

REMARK. If $p = 1$, we may take $\beta = 0$, obtaining

$$f(\lambda) = \int_0^\infty e^{-\lambda\xi} a_0(\xi)\, d\xi. \tag{10.6.3}$$

By analogy with the numerically-valued case, we would expect the same representation to hold also for $1 < p \leqq 2$ with an $a_0(\xi)$ such that $e^{-\alpha\xi}a_0(\xi) \in B_{1/p'}[(0, \infty); \mathfrak{X}]$. We do not know if this is actually so; since Bochner has shown that Bessel's inequality does not necessarily hold for functions in $B_2[S; \mathfrak{X}]$, the basis for a satisfactory Fourier transform theory seems to be lacking when $p > 1$ and this closes the usual avenue of approach to the Laplace transform theory as well.

THEOREM 10.6.2. *Let $f(\lambda)$ be holomorphic and of finite order ω in the half-plane $\sigma \geqq \alpha$. Then $a_\beta(\xi)$ exists as a continuous function of ξ for $\beta > \omega + 1$ and*

$$f(\lambda) = \lambda^\beta \int_0^\infty e^{-\lambda\xi} a_\beta(\xi)\, d\xi, \tag{10.6.4}$$

convergent for $\sigma >$ max $(0, \alpha)$.

The proof is analogous to that of the preceding theorem and is omitted.

We see, in particular, that if $f(\lambda)$ is holomorphic and of finite order in a given half-plane $\sigma \geqq \alpha \geqq 0$ and if $\sigma_1 > \alpha$, then formula (10.6.4) holds for $\sigma > \sigma_1$ provided $\beta > \mu(\sigma_1) + 1$. Increasing the value of β is a convergency preserving transformation which enables us to represent $f(\lambda)$ by Laplace integrals in the largest half-plane in which $f(\lambda)$ is holomorphic and of finite order. As applied to $a_\beta(\xi)$, this transformation amounts to a fractional integration. Indeed, we read off from formula (10.2.17) that

$$a_{\beta+\gamma}(\xi) = \frac{1}{\Gamma(\gamma)} \int_0^\xi (\xi - \tau)^{\gamma-1} a_\beta(\tau)\, dt. \tag{10.6.5}$$

3. Binomial Series

10.7. Properties of the series. Functions which are holomorphic and of exponential type in a half-plane play an important role in classical analysis. They are equally important in vector-valued function theory. Such functions admit of representations in terms of *binomial series*, also known as *binomial coefficient series* or *Newton's interpolation series*. We shall sketch the theory of such series here since they will be used extensively in Chapter XIII.

A binomial series is an expansion of the form

$$u_0 + u_1 \frac{\zeta}{1} + u_2 \frac{\zeta(\zeta-1)}{1 \cdot 2} + \cdots + u_n \frac{\zeta(\zeta-1)\cdots(\zeta-n+1)}{1 \cdot 2 \cdots n} + \cdots \tag{10.7.1}$$
$$\equiv \sum_{n=0}^{\infty} u_n \binom{\zeta}{n},$$

where in the vector-valued case the quantities u_n are elements of a complex (B)-space $\mathfrak{U}$. In the applications which we have in view, $\mathfrak{U}$ will be a space $\mathfrak{E}(\mathfrak{X})$, that is, the coefficients u_n will be linear bounded operators on a complex (B)-space $\mathfrak{X}$ to itself.

The series converges trivially when ζ is a non-negative integer since all but a finite number of terms vanish. If the series converges for $\zeta = \zeta_0$, where ζ_0 is not zero or a positive integer, then one shows with the aid of Abel's summation formula that it converges for every ζ with $\Re(\zeta) > \Re(\zeta_0)$. Moreover, the convergence is uniform with respect to ζ in every finite closed sector, having its vertex at $\zeta = \zeta_0$ but lying otherwise in the interior of the half-plane. Thus the region of convergence is a half-plane $\xi > \sigma_0$, $\zeta = \xi + i\eta$, to which has to be added the trivial points $\zeta = 0, 1, \cdots, [\sigma_0]$, and possibly also a point set on the line of convergence $\xi = \sigma_0$. Further, the sum of the series, $f(\zeta)$ say, is a function on complex numbers to $\mathfrak{U}$ which is holomorphic for $\xi > \sigma_0$ by virtue of the uniform convergence. Similarly, the region of absolute convergence is a half-plane $\xi > \sigma_a$ plus, possibly, the line of absolute convergence $\xi = \sigma_a$.

The abscissas of ordinary and of absolute convergence are given by

$$\sigma_0 = -1 + \limsup_{n \to \infty} \log \left\| \sum_{k=0}^{n} (-1)^k u_k \right\| \Big/ \log n, \tag{10.7.2}$$

$$\sigma_a = -1 + \limsup_{n \to \infty} \log \left[\sum_{k=0}^{n} \| u_k \| \right] \Big/ \log n, \tag{10.7.3}$$

whenever these limits are positive. If the first one is zero, but the series $\sum_0^\infty (-1)^k u_k$ diverges, σ_0 is still given by (10.7.2); if it converges instead, we have to replace

$$\sum_{0}^{n} (-1)^k u_k \quad \text{by} \quad \sum_{n+1}^{\infty} (-1)^k u_k$$

in the formula. Analogous changes have to be made in the second formula. The reader will have no difficulties in verifying these formulas; cf. N. E. Nörlund

[2, pp. 111–115] where $z - 1$ should be replaced by ζ and absolute values by norms in carrying over the proof. It is obvious from the formulas that

$$0 \leqq \sigma_a - \sigma_0 \leqq 1. \tag{10.7.4}$$

The expansion of a numerical function in binomial series is not unique owing to the presence of *null series*. If m is a positive integer, the series

$$\psi_m(\zeta) = \sum_{n=m}^{\infty} (-1)^{n+m} \binom{n}{m}\binom{\zeta}{n}$$

converges for $\xi > m$ to the sum zero. It has the same sum for $\zeta = 0, 1, \cdots m - 1$, and equals one for $\zeta = m$. This fact may be used to obtain the so-called *reduced series* for $f(\zeta)$. We shall suppose that $f(\zeta)$ is known in advance to be holomorphic in a right half-plane and to be defined at the origin and the positive integers. We can then find coefficients $\gamma_0, \gamma_1, \cdots, \gamma_p$, where $p = [\sigma_0]$, so that the function

$$f(\zeta) + \sum_{m=0}^{p} \gamma_m \psi_m(\zeta) f(m)$$

has a binomial expansion convergent for $\xi > \sigma_0$, the sum of the series being $f(m)$ for $\zeta = m, m = 0, 1, \cdots, p$. The resulting binomial series is known as the reduced series. In the following all binomial series will be supposed to be reduced series.

In the case of a reduced series, the coefficients u_n are uniquely determined by the values of $f(\zeta)$ at the non-negative integers. We find by substitution that

$$u_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k) = \Delta^n f(0), \qquad n = 0, 1, 2, \cdots. \tag{10.7.5}$$

The function $f(\zeta)$ may also be represented by a Laplace integral of the form

$$f(\zeta) = \frac{1}{2\pi i} \int_C g(\lambda)\lambda^{\zeta}\, d\lambda, \tag{10.7.6}$$

$$g(\lambda) = \sum_{n=0}^{\infty} \Delta^n f(0)(\lambda - 1)^{-n-1} = \sum_{n=0}^{\infty} f(n)\lambda^{-n-1}, \tag{10.7.7}$$

where the first series converges for $|\lambda - 1| > 1$ and the second for $|\lambda| > 2$. Further

$$\lim_{\lambda \to 0} |\lambda|^{\sigma_0 - 1 + \epsilon} \| g(\lambda) \| = 0 \tag{10.7.8}$$

for every $\epsilon > 0$, if $\lambda \to 0$ in the sector $\frac{1}{2}\pi + \delta \leqq \arg \lambda \leqq 3 \cdot \frac{1}{2}\pi - \delta$. Finally C is a closed rectifiable path, surrounding the circle $|\lambda - 1| = 1$ once in the positive sense, beginning and ending at the origin in the neighborhood of which the path lies in the sector just mentioned. The power has its principal determination, $\lambda^{\zeta} = \exp(\zeta \log \lambda)$, where the imaginary part of the logarithm lies between $-\pi$ and π. These assertions are easily verified by direct computation; see also section 13.6 where a special case is carried through in some detail.

The function $f(\zeta)$ defined by (10.7.1) for $\xi > \sigma_0$ is of exponential type in every interior half-plane. If $\beta > \sigma_0$ and if

$$h(\phi; f) = \limsup_{r \to \infty} r^{-1} \log \| f(\beta + re^{i\phi}) \| \tag{10.7.9}$$

is *the Phragmén-Lindelöf growth function* of $f(\zeta)$ defined for $-\frac{1}{2}\pi \leqq \phi \leqq \frac{1}{2}\pi$, then

$$h(\phi; f) \leqq l(\phi) \tag{10.7.10}$$

where

$$l(\phi) = \phi \sin \phi + \cos \phi \log (2 \cos \phi), \tag{10.7.11}$$

and $\log 2 \leqq l(\phi) \leqq \frac{1}{2}\pi$. More precisely, we have the estimate due to F. Carlson [2]

$$\| f(\beta + re^{i\phi}) \| \leqq \exp [rl(\phi)]\, r^{\beta + \frac{1}{2} + \epsilon(r)} (1 + r \cos \phi)^{-\frac{1}{2}}, \tag{10.7.12}$$

where $\epsilon(r)$ tends uniformly to zero when ζ tends to infinity in the sector. The proof goes through as in the numerical case, replacing absolute values by norms. The crux of the proof lies in showing that

$$\left| \binom{\zeta}{n} \right| < Cr^{\frac{1}{2}} \exp [rl(\phi)]$$

for all n when r is large and that the terms which have the greatest influence on the order of $f(\zeta)$ are those whose subscripts lie in a certain neighborhood of the critical value $r/(2 \cos \phi)$ for which the estimates of the binomial coefficients may be reversed. We do not insist on these details, but add the observation that $l(\phi)$ *is the function of support of the closed convex region*

$$e^{\xi} \leqq 2 \cos \eta, \tag{10.7.13}$$

which in its turn is the image of the circle $| \lambda - 1 | \leqq 1$ under the conformal mapping $\lambda = e^{\zeta}$.

10.8. Representation and analytic continuation. Conversely every function $f(\zeta)$ which is holomorphic for $\xi \geqq \beta$ and satisfies an estimate of the above type can be represented by a convergent binomial series. This is proved as in the numerical case, that is, using Newton's interpolation formula, expressing the remainder by Cauchy's integral and estimating its norm. See Nörlund (op. cit., p. 131 et seq.). The result may be formulated as follows:

THEOREM 10.8.1. *If $f(\zeta)$ is a function on complex numbers to $\mathfrak{U}$, holomorphic in the half-plane $\xi \geqq \beta$, where it satisfies the inequality*

$$\| f(\beta + re^{i\phi}) \| \leqq e^{rl(\phi)} (1 + r)^{\gamma + \epsilon(r)}, \qquad -\frac{\pi}{2} \leqq \phi \leqq \frac{\pi}{2}, \tag{10.8.1}$$

and the function $\epsilon(r)$ *tends uniformly to zero when* r *tends to infinity, then* $f(\zeta)$ *may be expanded in a binomial series of type* (10.7.1), *the abscissa of convergence of which does not exceed the larger of the two numbers* $\beta, \gamma + \frac{1}{2}$.

A numerical binomial series may be summed by the method of Cesàro which in the present case is equivalent to applying the transformation $\zeta' = \zeta + k$. The much more powerful transformation $\zeta' = \alpha\zeta$ has been used by Nörlund; it gives the analytic continuation of $f(\zeta)$ in the largest half-plane where it is holomorphic and of exponential type. This method applies also to vector-valued functions. Indeed, if $f(\zeta)$ is holomorphic in $\xi \geqq \beta$ and

$$||f(\beta + re^{i\phi})|| \leqq Ae^{Br}, \qquad -\frac{\pi}{2} \leqq \phi \leqq \frac{\pi}{2},$$

then we can clearly determine a positive α so small that $f(\alpha\zeta)$ satisfies (10.8.1) and is consequently representable by a convergent binomial series in the variable ζ. This implies a convergent binomial series in the variable ζ/α for the original function $f(\zeta)$. Since $\min l(\phi) = \log 2$, it suffices to take $\alpha \leqq (\log 2)/B$. In particular, if $f(\zeta)$ admits of a representation by a binomial series in ζ, then it is also representable by a binomial series in ζ/α for $0 < \alpha < 1$. Let $\sigma_0(\alpha)$ be the abscissa of convergence of the binomial series in ζ/α. Then $\sigma_0(\alpha)$ is a never decreasing function of α which is completely determined by the rate of growth of $||f(\zeta)||$ on vertical lines.

With Nörlund we introduce

$$\gamma(\xi) = \limsup_{|\eta|\to\infty} |\eta|^{-1} \log ||f(\xi + i\eta)||, \tag{10.8.2}$$

which is clearly well defined in the largest half-plane in which $f(\zeta)$ is holomorphic and of exponential type. Let ξ_0 be so chosen that $f(\zeta)$ is holomorphic and of finite exponential type for $\xi \geqq \xi_0 + \epsilon$ when $\epsilon > 0$, but lacks at least one of these properties for $\xi > \xi_0 - \epsilon$, no matter how small ϵ is. The function $\gamma(\xi)$ is defined for $\xi > \xi_0$ and has properties analogous to those of Lindelöf's function $\mu(\sigma)$ of formula (10.5.2); $\gamma(\xi)$ *is monotonic never increasing, convex, and continuous for* $\xi > \xi_0$, *but, in addition, it is non-negative.* All these properties follow from the Phragmén-Lindelöf extension of the principle of the maximum.

Let (α_0, α_1) be the largest interval such that the equation

$$\gamma(\sigma) = \frac{\pi}{2\alpha}$$

has a solution, which is necessarily unique, say $\sigma = \sigma_0(\alpha)$, when $0 \leqq \alpha_0 < \alpha < \alpha_1 \leqq \infty$. This is the required abscissa of convergence. For $0 < \alpha \leqq \alpha_0$ we have $\sigma_0(\alpha) = \sigma_0(\alpha_0)$, for $\alpha_1 < \alpha$, $\sigma_0(\alpha) = +\infty$. Further $\sigma_0(\alpha)$ is monotonic never decreasing and continuous for $\alpha_0 < \alpha < \alpha_1$. These assertions are proved as in the numerical case. For future reference we formulate the result as follows:

THEOREM 10.8.2. *Let $f(\zeta)$ be a function on complex numbers to $\mathfrak{U}$ which is holomorphic and of exponential type for $\xi \geqq \xi_0 + \epsilon$ when $\epsilon > 0$, but lacks at least one of these properties for $\xi > \xi_0 - \epsilon$, no matter how small ϵ is. If $\xi_0 > 0$, we assume in addition that $f(\zeta)$ is defined for all real non-negative values of ζ. Let $\gamma(\xi)$, $\sigma_0(\alpha)$, α_0, and α_1 have the meaning defined above. Then*

$$f(\zeta) = \sum_{n=0}^{\infty} \Delta_\alpha^n f(0) \frac{1}{n!} \zeta(\zeta - \alpha) \cdots (\zeta - (n-1)\alpha), \tag{10.8.3}$$

where the series converges for $\xi > \xi_0$ when $0 < \alpha \leqq \alpha_0$, and for $\xi > \sigma_0(\alpha)$ when $\alpha_0 < \alpha < \alpha_1$. The series fails to converge for any non-trivial value of ζ when $\alpha_1 < \alpha$.

It should be observed that the values assigned to $f(\zeta)$ on the interval $[0, \xi_0]$ are actually immaterial. Changing these values merely adds a null series to (10.8.3) and does not change the abscissas of convergence.

CHAPTER XI
GENERATOR AND RESOLVENT

11.1. Orientation. Further study of the semi-group $\mathfrak{S} = \{T(\xi)\}$ of linear bounded transformations on a complex (B)-space to itself centers around the properties of the *generating transformation* A and its *resolvent* $R(\lambda; A)$. The Laplace transform turns out to be the natural intermediary between $T(\xi)$ and $R(\lambda; A)$.

The following simple case will elucidate the situation. We take $\mathfrak{X} = Z_1$, the space of complex numbers with the usual metric, and consider linear transformations on Z_1 to itself. Here

$$T(\xi)\zeta = e^{\alpha\xi}\zeta, \qquad 0 < \xi < \infty,$$

defines a one-parameter semi-group of linear transformations, the generating transformation being

$$A\zeta = \alpha\zeta$$

with the resolvent

$$R(\lambda; A)\zeta = \frac{1}{\lambda - \alpha}\zeta.$$

The function $(\lambda - \alpha)^{-1}$ is the Laplace transform of $e^{\alpha\xi}$

$$\int_0^\infty e^{-\lambda\xi} e^{\alpha\xi}\, d\xi = \frac{1}{\lambda - \alpha}$$

for $\Re(\lambda) > \Re(\alpha)$. Conversely

$$\frac{1}{2\pi i}\int_C e^{\lambda\xi}\frac{d\lambda}{\lambda - \alpha} = e^{\alpha\xi}$$

for a suitable choice of the path of integration.

Using the heuristic *correspondence principle* $\alpha \to A$, then $e^{\alpha\xi} \to T(\xi)$, $(\lambda - \alpha)^{-1} \to R(\lambda; A)$ and we are led to the relations

$$\int_0^\infty e^{-\lambda\xi}\, T(\xi)\, d\xi = R(\lambda; A), \tag{11.1.1}$$

$$\frac{1}{2\pi i}\int e^{\lambda\xi} R(\lambda; A)\, d\lambda = T(\xi). \tag{11.1.2}$$

Thus *we would expect the resolvent of the generator to be the Laplace transform of the semi-group operator* $T(\xi)$, *conversely the latter should be obtainable from the resolvent by the inversion of the Laplace integral.*

These expectations turn out to be fully justified but there is of course the usual difference between the uniform and the strong case when it comes to the interpretation of the formulas. The fact that $R(\lambda; A)$ is holomorphic at infinity in the uniform case (and only in this case) simplifies its discussion.

The relations between $R(\lambda; A)$ and $T(\xi)$ provide the subject matter of the present chapter. There are two paragraphs corresponding to the uniform and the strong cases respectively.

References. Fukamiya [1], Gelfand [3], Hille [9], Stone [1].

1. The Uniform Case

11.2. The resolvent. We suppose that $\mathfrak{B}$ is a complex Banach algebra with unit element e. Let $f(\xi)$ be a measurable function on $(0, \infty)$ to $\mathfrak{B}$ such that for all ξ_1, ξ_2

$$f(\xi_1 + \xi_2) = f(\xi_1)f(\xi_2).$$

It was shown in Theorem 8.3.1 that $f(\xi)$ is necessarily continuous for $\xi > 0$ but need not approach any limit when $\xi \to 0$. Under the additional assumption that $\lim_{\xi\to 0} f(\xi) = e$, it was shown in Theorem 8.4.2 that there exists an $a \in \mathfrak{B}$ such that

$$f(\xi) = \exp(\xi a) = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} a^n$$

and this function is also defined and satisfies the functional equation for complex values of the scalar variable.

We may consequently restrict ourselves to a study of

$$f(\zeta) = \exp(\zeta a), \qquad a \in \mathfrak{B}. \tag{11.2.1}$$

The assumption that $f(0) = e$ rather than an arbitrary idempotent $j \neq e$ is natural since we want to consider the resolvent of a. Otherwise we have to restrict ourselves to the subalgebra $j\mathfrak{B}j$ in which j plays the role of unit element.

It is clear that

$$\limsup_{\omega\to\infty} \frac{1}{\omega} \log \| \exp(\omega a) \| \leqq \| a \|$$

so that the Laplace integral (note that $e^{-\lambda\xi} = \exp(-\lambda\xi)$)

$$R(\lambda) = \int_0^\infty e^{-\lambda\xi} \exp(\xi a)\, d\xi \tag{11.2.2}$$

exists and is absolutely convergent for $\Re(\lambda) > \| a \|$. For such values of λ we may compute the value of the integral by substituting the power series of

$\exp(\xi a)$ and integrating term-wise. The result is

$$R(\lambda) = \sum_{n=0}^{\infty} a^n \lambda^{-n-1} = R(\lambda; a) = (\lambda e - a)^{-1}. \tag{11.2.3}$$

This function is of course holomorphic for $|\lambda| > \|a\|$ and the spectrum of a is located inside the circle $|\lambda| = \|a\|$. Since $\exp(\zeta a)$ is an entire function of order one and type $\leqq \|a\|$ in the sense of section 3.12, the integral in (11.2.2) may be taken along an arbitrary ray $\arg \zeta = \varphi$ instead of along the real axis in the ζ-plane. The integral will then converge absolutely for $\Re[\lambda \exp(i\varphi)] > \|a\|$ and represents $R(\lambda; a)$ in this half-plane. We have thus proved:

THEOREM 11.2.1. *If $a \in \mathfrak{B}$ is the generating element of $f(\zeta)$ so that $f(\zeta) = \exp(\zeta a)$, then $R(\lambda; a)$, the resolvent of a, is the Laplace transform of $f(\zeta)$.*

11.3. Inversion of the resolvent. We can invert this Laplace transform using any one of the methods developed in section 10.3. Since $R(\lambda; a)$ is holomorphic at infinity, the situation is more favorable than in the general case and the inversion formulas give more. The best result is given by a formula which does not have a meaning in the general case.

THEOREM 11.3.1. *If Γ is a simple closed rectifiable curve surrounding the spectrum of a in the positive sense then*

$$\exp(\zeta a) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda\zeta} R(\lambda; a)\, d\lambda. \tag{11.3.1}$$

PROOF. Γ may be deformed into a circle $|\lambda| = \rho > \|a\|$. Substituting the power series in $1/\lambda$ for $R(\lambda; a)$ and integrating term-wise, the power series for $\exp(\zeta a)$ is obtained. We note that (11.3.1) represents $\exp(\zeta a)$ for all complex values of ζ.

The information obtained from Theorem 10.3.2 is not quite so good. To start with we see that for $\gamma > \|a\|$, ζ real positive

$$\exp(\zeta a) = \frac{1}{2\pi i} (C, 1) \text{-} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda\zeta} R(\lambda; a)\, d\lambda.$$

Actually, however, the integral exists in the Cauchy sense and we have even

$$\exp(\zeta a) = e + \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda\zeta} \left[R(\lambda; a) - \frac{e}{\lambda} \right] d\lambda, \tag{11.3.2}$$

where the latter integral is absolutely convergent since the expression in square brackets is $O(|\lambda|^{-2})$. Here we have of course made use of the fact that the $(C, 1)$-limit coincides with the Cauchy limit when the latter exists. The formula can be made to represent $\exp(\zeta a)$ on the ray $\arg \zeta = \varphi$ by turning the line of integration so that it becomes perpendicular to the direction $\arg \lambda = -\varphi$. If use is made of this artifice, formula (11.3.2) has the same range of power as

(11.3.1). The latter is of course a special case of the abstract analog of the classical relation between an entire function of exponential type and its so-called *Borel transform.*

From Theorem 10.3.3 we conclude that

$$\exp(\zeta a) = \lim_{\omega \to \infty} \omega \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n-1)!} R(n\omega; a) \exp(n\omega\zeta), \tag{11.3.3}$$

where ω tends to infinity in such a manner that $\omega\zeta$ stays real and positive. We get relations of more fundamental nature using Widder's operator

$$L_{k,\zeta}[R(\lambda; a)] = \frac{(-1)^k}{k!} \left(\frac{k}{\zeta}\right)^{k+1} R^{(k)}\left(\frac{k}{\zeta}; a\right)$$

which reduces to

$$L_{k,\zeta}[R(\lambda; a)] = \left[\frac{k}{\zeta} R\left(\frac{k}{\zeta}; a\right)\right]^{k+1}$$

by formula (5.8.3). Since $\lambda R(\lambda; a) \to e$ when $|\lambda| \to \infty$, Theorem 10.3.4 leads to the following important relation

$$\exp(\zeta a) = \lim_{k \to \infty} \left[\frac{k}{\zeta} R\left(\frac{k}{\zeta}; a\right)\right]^k, \tag{11.3.4}$$

which may be rewritten in the more suggestive form

$$\exp(\zeta a) = \lim_{k \to \infty} \left[e - \frac{\zeta}{k} a\right]^{-k} \tag{11.3.5}$$

and this representation is valid for all complex values of ζ. In order to see this we note that

$$\begin{aligned} L_{k,\zeta}[R(\lambda; a)] &= \frac{k}{\zeta k!} \int_0^\infty \exp\left(-\frac{k\tau}{\zeta}\right)\left(\frac{k\tau}{\zeta}\right)^k \exp(\tau a)\, d\tau \\ &= \frac{1}{\zeta} \int_0^\infty W_0\left(\left|\frac{\tau}{\zeta}\right|; k\right) \exp(\tau a)\, d\tau \end{aligned}$$

if the integral is taken along the ray $\arg \tau = \arg \zeta$ as is permitted. Since $d\tau/\zeta$ is real positive, this is Widder's singular integral and the conclusion of Theorem 10.3.4 is valid. This argument shows that one more of the classical definitions of the exponential function extends to the abstract case:

THEOREM 11.3.2. *For any element b of $\mathfrak{B}$ we have*

$$\exp(b) = \lim_{k \to \infty} \left[e - \frac{1}{k} b\right]^{-k}. \tag{11.3.6}$$

Actually a direct convergence proof shows that

$$\exp(b) = \lim_{k \to \infty} \left[e + \frac{1}{k} b\right]^k \tag{11.3.7}$$

is true. It turns out, however, that in operator theory the first of these definitions is preferable to the second inasmuch as the first one can be used to assign a meaning to $\exp(U)$ for certain classes of unbounded operators U while the second is useless for this purpose.

Suppose now that a is an element of $\mathfrak{B}$ such that $R(\lambda; a)$ exists for $\lambda > 0$ and $\lambda \| R(\lambda; a) \| \leqq 1$ for such values of λ. Since $R(\lambda; a)$ is holomorphic at infinity, Theorem 11.3.1 applies and shows that $R(\lambda; a)$ is the Laplace transform of $\exp(\zeta a)$. The boundedness assumption shows that $\lambda^{k+1} \| R^{(k)}(\lambda; a) \| \leqq 1$ for $k = 0, 1, 2, \cdots$ and all $\lambda > 0$. Hence by Theorem 10.3.5 $\| \exp(\xi a) \| \leqq 1$ for all $\xi > 0$. Further $R(\lambda; a)$ is holomorphic for $\Re(\lambda) = \sigma > 0$ and

$$\sigma \| R(\sigma + i\tau; a) \| \leqq 1, \qquad \sigma > 0. \tag{11.3.8}$$

It is clear that the (bounded) spectrum of a is located in $\sigma \leqq 0$.

11.4. Analytical one-parameter groups of linear transformations. The most important instance of the preceding theory is that in which $\mathfrak{B} = \mathfrak{E}(\mathfrak{X})$, the space of linear bounded transformations on a complex (B)-space $\mathfrak{X}$ to itself, and $\mathfrak{G} = \{T(\zeta)\}$ is a one-parameter group of such transformations with

$$T(\zeta_1 + \zeta_2) = T(\zeta_1)T(\zeta_2), \; T(0) = I. \tag{11.4.1}$$

The discussion in sections 8.4, 11.2, and 11.3 leads to the following result.

THEOREM 11.4.1. *If the operator $T(\xi) \in \mathfrak{E}(\mathfrak{X})$ is defined for $\xi > 0$ and satisfies* (11.4.1) *for real positive values of the parameter and if $\| T(\xi) - I \| \to 0$ with ξ, then there exists an operator $A \in \mathfrak{E}(\mathfrak{X})$ such that $T(\xi) = \exp(\xi A)$ for $\xi > 0$. Defining $T(\zeta) = \exp(\zeta A)$ for all complex ζ, then $\mathfrak{G} = \{T(\zeta)\}$ is an analytical group. The resolvent of A is the Laplace transform of $T(\zeta)$,*

$$R(\lambda; A) = \int_0^\infty e^{-\lambda\zeta} T(\zeta)\, d\zeta, \tag{11.4.2}$$

where the integral may be taken along the ray $\arg \zeta = -\arg \lambda$ *and the representation is valid at least for $|\lambda| > \| A \|$. Further*

$$T(\zeta) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda\zeta} R(\lambda; A)\, d\lambda, \tag{11.4.3}$$

Γ surrounding the spectrum of A in the positive sense, and also

$$T(\zeta) = \lim_{k \to \infty} \left\{ \frac{k}{\zeta} R\left(\frac{k}{\zeta}; A\right) \right\}^k. \tag{11.4.4}$$

Conversely, each element A of $\mathfrak{E}(\mathfrak{X})$ defines a one-parameter analytical group $\mathfrak{G} = \{\exp(\zeta A)\}$ and this group coincides with the family of operators defined in terms of the resolvent of A by formulas (11.4.3) *and* (11.4.4).

COROLLARY. *A semi-group $\mathfrak{S} = \{T(\xi)\}$, $\xi > 0$, has a bounded infinitesimal generator if and only if $T(\xi) \to I$ in the uniform topology when $\xi \to 0$. Such a semi-group can always be embedded in an analytical one-parameter group.*

2. The Strong Case

11.5. Further properties of the generator. We assume that $\mathfrak{X}$ is a complex (B)-space, $\mathfrak{S} = \{T(\xi)\}$ a semi-group of linear bounded transformations on $\mathfrak{X}$ to itself defined for $\xi > 0$. We know that assuming $T(\xi)$ to be strongly measurable and $\| T(\xi) \|$ to be bounded on each interval $(\epsilon, 1/\epsilon)$ is equivalent to $T(\xi)$ being strongly continuous for $\xi > 0$ but does not imply that $T(\xi)$ tends to a strong limit when $\xi \to 0$ or that $\| T(\xi) \|$ is bounded on $(0, 1)$. Thus additional assumptions are needed if a simple theory is desired. Such conditions are listed in Theorem 9.4.1 or, what is equivalent, we may impose the single assumption:

C_0. *$T(\xi)$ converges strongly but not uniformly to I when $\xi \to 0$.*

We exclude the possibility of $T(\xi)$ tending uniformly to I as this case was taken care of in the preceding paragraph. We know that the assumption C_0 implies that $T(\xi)$ is strongly continuous for $\xi > 0$. Further $\| T(\xi) \|$ is bounded in every finite interval $(0, \omega)$. The infinitesimal generator A of $\mathfrak{S}$ is now an unbounded transformation which is closed by Theorem 9.6.3. Its domain of definition $\mathfrak{D}[A]$ is dense in $\mathfrak{X}$ and so is $\mathfrak{D}[A^n]$ for every positive integral n by Theorem 9.5.4. We shall now prove a sharper result:

Theorem 11.5.1. *$\bigcap_n \mathfrak{D}[A^n]$ is dense in $\mathfrak{X}$.*

Proof. We use a construction due to I. Gelfand for the case of one-parameter groups. Let $\mathfrak{K}$ be the class of all numerical functions $K(\tau)$ with the following properties:

(i) $K(\tau)$ has derivatives of all orders in $(0, \infty)$;
(ii) $K^{(n)}(\tau) \to 0$ when $\tau \to 0$, $n = 0, 1, 2, \cdots$;
(iii) for every $\gamma \geqq 0$ and $n \geqq 0$, $e^{\gamma\tau}K^{(n)}(\tau) \in L(0, \infty)$.

We consider the set $\mathfrak{X}[\mathfrak{K}]$ of all elements y of the form

$$y = \int_0^\infty K(\tau)\, T(\tau)x \, d\tau, \qquad x \in \mathfrak{X}. \tag{11.5.1}$$

Since $\| T(\tau)x \| \leqq Me^{\gamma\tau}$ for some finite M and γ, we see that the integral exists.

$$\begin{aligned} A_\eta y &= \frac{1}{\eta} \int_0^\infty K(\tau)[T(\tau + \eta) - T(\tau)]x \, d\tau \\ &= \int_\eta^\infty \frac{1}{\eta} [K(\tau - \eta) - K(\tau)]T(\tau)x \, d\tau - \frac{1}{\eta} \int_0^\eta K(\tau)T(\tau)x \, d\tau \\ &\to - \int_0^\infty K'(\tau)T(\tau)x \, d\tau \qquad \text{when } \eta \to 0, \end{aligned}$$

since the difference quotient converges in the mean to $K'(\tau)$ while the second term in the third member tends to θ by (ii). Repeating the argument we see that $A^n y$ exists for each n and

$$A^n y = (-1)^n \int_0^\infty K^{(n)}(\tau)T(\tau)x \, d\tau. \tag{11.5.2}$$

We want to show that $\mathfrak{X}[\mathfrak{K}]$ is dense in $\mathfrak{X}$. The converse assumption implies the existence of a bounded linear functional x^* $(\neq \theta)$ such that $x^*\{\mathfrak{X}[\mathfrak{K}]\} = 0$. Among the functions $K(\tau)$ in $\mathfrak{K}$ we note $\exp(-k\tau^2 - 1/\tau)$. It follows then that

$$\int_0^\infty e^{-k\tau^2} e^{-1/\tau} x^*[T(\tau)x]\, d\tau = 0$$

for all $x \in \mathfrak{X}$ and $k = 1, 2, 3, \cdots$. By Lerch's theorem this implies that $x^*[T(\tau)x] \equiv 0$, $\tau > 0$. Since $T(\tau)x \to x$ when $\tau \to 0$ and x^* is continuous, this gives $x^*[x] = 0$ for all x so that x^* has to be the zero functional. This contradiction shows that $\mathfrak{X}[\mathfrak{K}]$ is dense in $\mathfrak{X}$. But $\mathfrak{X}[\mathfrak{K}] \subset \bigcap_n \mathfrak{D}[A^n]$ so the theorem is proved.

If $y \in \mathfrak{X}[\mathfrak{K}]$, then $T(\xi)y$ has derivatives of all orders with respect to ξ. This does not imply that $T(\xi)y$ is analytic in ξ, however. The method used by Gelfand in the group case in order to bridge the gap from indefinite differentiability to analyticity does not seem to work for semi-groups. In the group case the set of elements y such that $T(\xi)y$ is an analytic function of ξ is dense in $\mathfrak{X}$; we do not know if this is true also for continuous semi-groups.

11.6. The resolvent. Following the prescript of section 11.2 we form the integral

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda\xi} T(\xi)x\, d\xi \tag{11.6.1}$$

and prove

THEOREM 11.6.1. *Let*

$$\alpha = \lim_{\omega\to\infty} \frac{1}{\omega} \log \| T(\omega) \|. \tag{11.6.2}$$

Then (11.6.1) *converges for* $\Re(\lambda) > \alpha$ *regardless of* $x \in \mathfrak{X}$. *For each such* λ *the integral defines a linear bounded transformation on* $\mathfrak{X}$ *to itself with the following properties*:

(1) $(\lambda I - A)R(\lambda; A)x = x$ *for each* $x \in \mathfrak{X}$, $R(\lambda; A)(\lambda I - A)x = x$ *for each* $x \in \mathfrak{D}[A]$;

(2) *for fixed* λ *the range of* $R(\lambda; A)$ *is dense in* $\mathfrak{X}$;

(3) $R(\lambda; A)x = \theta$ *for a fixed* λ *implies* $x = \theta$.

PROOF. The abscissa of absolute convergence of (11.6.1) is given by

$$\sigma_a(x) = \limsup_{\omega\to\infty} \frac{1}{\omega} \log \| T(\omega)x \| \leqq \alpha,$$

which proves the first assertion. It is not true in general that $\sigma_a(x) = \alpha$ for all x; even the relation $\sup_{\|x\|=1} \sigma_a(x) = \alpha$ is in doubt. An any rate, the integral defines a holomorphic function of λ in the half-plane $\Re(\lambda) > \alpha$ regardless of x.

If β is any fixed number, $\beta > \alpha$, we can find an $M = M(\beta)$ such that $\| T(\xi) \| \leq Me^{\beta\xi}$ for $\xi \geqq 0$. It follows that

$$\| R(\lambda; A)x \| \leqq M[\Re(\lambda) - \beta]^{-1} \| x \|, \qquad \Re(\lambda) > \beta.$$

Thus the integral defines a linear bounded transformation on $\mathfrak{X}$ to itself for $\Re(\lambda) > \alpha$ and the norm is uniformly bounded with respect to λ for $\Re(\lambda) \geqq \alpha + \epsilon$.

We show next that $R(\lambda; A)x \in \mathfrak{D}[A]$ for all x if $\Re(\lambda) > \alpha$. Indeed

$$\begin{aligned} A_\eta R(\lambda; A)x &= \frac{1}{\eta}\int_0^\infty e^{-\lambda\xi}[T(\xi + \eta)x - T(\xi)x]\, d\xi \\ &= \frac{1}{\eta}(e^{\lambda\eta} - 1)\int_\eta^\infty e^{-\lambda\xi}T(\xi)x\, d\xi - \frac{1}{\eta}\int_0^\eta e^{-\lambda\xi}T(\xi)x\, d\xi \\ &\to \lambda R(\lambda; A)x - x \end{aligned}$$

when $\eta \to 0$ since $T(\eta)x \to x$. Hence $AR(\lambda; A)x$ always exists and we have verified the first relation under (1). Next, if $x \in \mathfrak{D}[A]$ Theorem 9.3.2 shows that

$$\begin{aligned} R(\lambda; A)Ax &= \int_0^\infty e^{-\lambda\xi}T(\xi)[Ax]\, d\xi = \int_0^\infty e^{-\lambda\xi}\frac{d}{d\xi}[T(\xi)x]\, d\xi \\ &= \lim_{\omega\to\infty}[e^{-\lambda\xi}T(\xi)x]_0^\omega + \lambda\int_0^\infty e^{-\lambda\xi}T(\xi)x\, d\xi \\ &= -x + \lambda R(\lambda; A)x. \end{aligned}$$

This proves the second relation under (1) and shows that $R(\lambda; A)$ is really the resolvent of A as indicated by the notation.

It was shown in section 5.14 that the resolvent of a linear closed unbounded transformation satisfies the first resolvent equation. Hence

(11.6.3) $$R(\lambda; A)x - R(\mu; A)x = (\mu - \lambda)R(\lambda; A)[R(\mu; A)x]$$

for all λ and μ whose real parts exceed α. This can also be verified by direct computation from the definition of $R(\lambda; A)$ as a Laplace integral.

Assuming λ fixed, $\Re(\lambda) > \alpha$, we consider the range of $R(\lambda; A)$. If it were not dense in $\mathfrak{X}$, then there would exist a linear bounded functional $x^*(\neq \theta)$ such that $x^*[R(\lambda; A)x] = 0$ for all x. From (11.6.3) one then concludes that $x^*[R(\mu; A)x] = 0$ for all x and all μ, $\Re(\mu) > \alpha$. This asserts that

$$\int_0^\infty e^{-\mu\xi}x^*[T(\xi)x]\, d\xi \equiv 0$$

as a function of μ for all x. By Lerch's theorem, $x^*[T(\xi)x]$, being a continuous function of ξ, must vanish identically for all ξ. In particular $x^*[x] = 0$ for all x. From this contradiction we conclude that (2) holds.

Conclusion (3) is proved in a similar manner. If $R(\lambda; A)x = \theta$ for a particular

choice of λ and x, then (11.6.3) shows that $R(\mu; A)x = \theta$ for all μ, $\Re(\mu) > \alpha$; by Theorem 10.3.2 this implies $T(\xi)x = \theta$ for all $\xi \geqq 0$. In particular, $x = \theta$. This completes the proof.

Since A is unbounded, $\sigma(A)$, the spectrum of A, is an unbounded point set. The only additional assertion that can be made is that $\sigma(A)$ lies in $\Re(\lambda) \leqq \alpha$. Actually simple examples can be adduced showing that every point of this half-plane may belong to $\sigma(A)$ (see section 16.2). Since $\sigma(A)$ is unbounded, $R(\lambda; A)$ cannot be holomorphic at infinity and the power series in $1/\lambda$ which represents $R(\lambda; T)$ in the case of a bounded operator T has no analog in the present case. Nevertheless, the series exists in a certain asymptotic sense on certain subspaces dense in $\mathfrak{X}$. More precisely, we have the following

THEOREM 11.6.2. *We have for every x in $\mathfrak{X}$*

$$\lim \lambda R(\lambda; A)x = x \tag{11.6.4}$$

if $|\lambda| \to \infty$, $|\arg \lambda| < \frac{1}{2}\pi$. *If* $x \in \mathfrak{D}[A^n]$

$$R(\lambda; A)x = \sum_{k=0}^{n-1} A^k x \lambda^{-k-1} + \lambda^{-n} R(\lambda; A)A^n x. \tag{11.6.5}$$

If $x \in \bigcap_n \mathfrak{D}[A^n]$ *this expansion is valid for every* n.

PROOF. We have

$$\lambda R(\lambda; A)x - x = \lambda \int_0^\infty e^{-\lambda\xi}[T(\xi)x - x]\, d\xi.$$

We shall prove that this difference tends to θ when $\lambda \to \infty$, uniformly with respect to λ in any fixed sector $|\arg \lambda| \leqq \varphi < \frac{1}{2}\pi$. To any given $\epsilon > 0$, we can find a $\delta = \delta(\epsilon, x)$ such that $\| T(\xi)x - x \| \leqq \epsilon$, $0 \leqq \xi \leqq \delta$. Further, to any $\beta > \alpha$ we can find an $M(\beta)$ such that $\| T(\xi) \| \leqq M(\beta)e^{\beta\xi}$, $\xi \geqq 0$. Putting $\Re(\lambda) = \sigma$ we have

$$\left\| \lambda \int_0^\delta e^{-\lambda\xi}[T(\xi)x - x]\, d\xi \right\| \leqq \frac{|\lambda|}{\sigma}\epsilon \leqq \epsilon \operatorname{cosec} \varphi,$$

$$\begin{aligned} \left\| \lambda \int_\delta^\infty e^{-\lambda\xi}[T(\xi)x - x]\, d\xi \right\| &\leqq |\lambda| e^{-\delta\sigma} \left\{ \frac{M(\beta)}{\sigma - \beta} e^{\beta\delta} + \frac{1}{\sigma} \right\} \| x \| \\ &\leqq M_1(\beta) \operatorname{cosec} \varphi\, e^{-\frac{1}{2}|\lambda|\delta \cos \varphi} \| x \| \end{aligned}$$

if $\sigma \geqq 2\beta$. These estimates prove the first assertion.

If $x \in \mathfrak{D}[A^n]$ the second relation under (1) in Theorem 11.6.1 gives

$$R(\lambda; A)A^k x = \frac{1}{\lambda} \{A^k x + R(\lambda; A)A^{k+1}x\}, \qquad k = 0, 1, \cdots, n-1.$$

Successive substitution gives (11.6.5). In partiular, if $x \in \bigcap_n \mathfrak{D}[A^n]$ then (11.6.5) holds for all n. We recall that this point set is dense in $\mathfrak{X}$.

Since $\lim_{\lambda\to\infty} \lambda R(\lambda; A)A^n x = A^n x$ by (11.6.4) we have the following

THEOREM 11.6.3. *If* $x \in \bigcap_n \mathfrak{D}[A^n]$ *and* $\mathfrak{R}(\lambda) > \alpha$, *then*

$$R(\lambda; A)x \sim \sum_{n=0}^{\infty} A^n x \lambda^{-n-1}, \tag{11.6.6}$$

the series being asymptotic to $R(\lambda; A)x$ *in the sense of Poincaré, that is, for every* n

$$\lim_{\lambda \to \infty} \lambda^{n+1}[R(\lambda; A)x - \sum_{k=0}^{n-1} A^k x \lambda^{-k-1}] = A^n x, \tag{11.6.7}$$

where the limit exists uniformly with respect to λ *in any fixed sector* $|\arg \lambda| \leqq \varphi < \frac{1}{2}\pi$.

11.7. Inversion of the resolvent. We now proceed to the problem of expressing the semi-group operator $T(\xi)$ in terms of the resolvent $R(\lambda; A)$ of the generating transformation when $T(\xi)$ is continuous in the strong sense. There is no analog of Theorem 11.3.1 in this case, but we can apply all the methods developed in section 10.3. Theorem 10.3.1 gives the following result:

THEOREM 11.7.1. *For every* $x \in \mathfrak{X}$, $\xi \geqq 0$, *and* $\gamma > \max(0, \alpha)$

$$\int_0^\xi T(\tau)x \, d\tau = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda\xi} R(\lambda; A)x \frac{d\lambda}{\lambda}, \tag{11.7.1}$$

the limit existing uniformly with respect to ξ *in any finite interval. For every* $x \in \mathfrak{D}[A]$, $\xi > 0$, *we have*

$$T(\xi)x = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda\xi} R(\lambda; A)x \, d\lambda, \tag{11.7.2}$$

the limit existing uniformly with respect to ξ *in any interval* $(\epsilon, 1/\epsilon)$. *For* $\xi = 0$ *the limit is* $\frac{1}{2}x$.

PROOF. In applying the results of section 10.3 to the present case, we set $g(\xi) = T(\xi)x$, $a(\xi) = \int_0^\xi T(\tau)x \, d\tau$. The former is a continuous, the latter an absolutely continuous function of ξ. Formula (11.7.1) is then an immediate application of Theorem 10.3.1. For the second assertion we use formula (9.6.3) according to which

$$\int_0^\xi T(\tau)Ax \, d\tau = T(\xi)x - x,$$

together with the relation

$$R(\lambda; A)Ax = \lambda R(\lambda; A)x - x.$$

Substitution in (11.7.1) gives

$$T(\xi)x - x = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda\xi} R(\lambda; A)x \, d\lambda - x \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda\xi} \frac{d\lambda}{\lambda}.$$

The second limit is x if $\xi > 0$ but $\frac{1}{2}x$ if $\xi = 0$ and the limit exists uniformly with respect to ξ in $(\epsilon, 1/\epsilon)$. This completes the proof.

As an immediate consequence of Theorem 10.3.2 we obtain

Theorem 11.7.2. *For every $x \in \mathfrak{X}$, $\xi > 0$, and $\gamma > \max(0, \alpha)$*

$$T(\xi)x = \frac{1}{2\pi i}\,(C, 1)\text{-}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda\xi}R(\lambda; A)x\, d\lambda, \tag{11.7.3}$$

the limit existing uniformly with respect to ξ in every interval $(\epsilon, 1/\epsilon)$. For $\xi = 0$ the limit equals $\frac{1}{2}x$.

Before leaving the complex inversion formulas we shall derive the duals of Theorems 11.6.2 and 11.6.3. The first is a form of *Taylor's theorem with remainder*.

Theorem 11.7.3. *If $x \in \mathfrak{D}[A^n]$ and $\xi > 0$*

$$T(\xi)x = \sum_{k=0}^{n-1} \frac{\xi^k}{k!} A^k x + \frac{1}{(n-1)!}\int_0^\xi (\xi - \tau)^{n-1} T(\tau)A^n x\, d\tau. \tag{11.7.4}$$

Proof. We obtain this expansion by substituting (11.6.5) into (11.7.2). Formula (10.3.9) takes care of the remainder term and shows that

$$\begin{aligned}\lim_{\omega\to\infty} \frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega} & e^{\lambda\xi} R(\lambda; A)A^n x \frac{d\lambda}{\lambda^n} \\ &= \frac{1}{(n-2)!}\int_0^\xi (\xi - \tau)^{n-2}\int_0^\tau T(\eta)A^n x\, d\eta\, d\tau \\ &= \frac{1}{(n-1)!}\int_0^\xi (\xi - \tau)^{n-1}\, T(\tau)A^n x\, d\tau.\end{aligned}$$

A more direct and elementary proof is obtained by repeated integration by parts in the formula

$$T(\xi)x - x = \int_0^\xi T(\tau)Ax\, d\tau$$

observing that

$$\frac{d^k}{d\xi^k}[T(\xi)x] = T(\xi)A^k x.$$

The dual of Theorem 11.6.3 reads as follows:

Theorem 11.7.4. *If $x \in \bigcap_n \mathfrak{D}[A^n]$ and $\xi > 0$, then*

$$T(\xi)x \sim \sum_{n=0}^{\infty} \frac{\xi^n}{n!} A^n x, \tag{11.7.5}$$

the series being asymptotic in the sense of Poincaré to $T(\xi)x$ on the positive real axis, that is, for every n

$$\lim_{\xi\to 0} \xi^{-n}\left[T(\xi)x - \sum_{k=0}^{n-1} \frac{\xi^k}{k!} A^k x\right] = \frac{1}{n!} A^n x. \tag{11.7.6}$$

The verification of

$$\lim_{\xi\to 0} n\xi^{-n} \int_0^\xi (\xi - \tau)^{n-1} T(\tau) A^n x \, d\tau = A^n x$$

is left to the reader.

THEOREM 11.7.5. *For every $x \in \mathfrak{X}$ and $\xi > 0$*

$$T(\xi)x = \lim_{k\to\infty} \left[\frac{k}{\xi} R\left(\frac{k}{\xi}; A\right)\right]^k x, \tag{11.7.7}$$

an alternate form of which is

$$T(\xi)x = \lim_{k\to\infty} \left[I - \frac{\xi}{k} A\right]^{-k} x. \tag{11.7.8}$$

The limits exist uniformly with respect to ξ in every interval of the form $(\epsilon, 1/\epsilon)$.

PROOF. As in the case of (11.3.4) we find that

$$L_{k,\xi}[R(\lambda; A)x] = \left[\frac{k}{\xi} R\left(\frac{k}{\xi}; A\right)\right]^{k+1} x$$

and by Theorem 10.3.4 this expression tends to $T(\xi)x$ when $k \to \infty$ uniformly with respect to ξ in $(\epsilon, 1/\epsilon)$. The lowering of the exponent from $k + 1$ to k requires an estimate of the operators involved. We have

$$\begin{aligned}
[\lambda R(\lambda; A)]^k y &= \frac{(-1)^{k-1}\lambda^k}{(k-1)!} R^{(k-1)}(\lambda; A)y \\
&= \frac{\lambda^k}{(k-1)!} \int_0^\infty e^{-\lambda\tau} \tau^{k-1} T(\tau) y \, d\tau \\
&= \frac{1}{(k-1)!} \int_0^\infty e^{-\sigma} \sigma^{k-1} T\left(\frac{\sigma}{\lambda}\right) y \, d\sigma.
\end{aligned}$$

In view of the estimate $\| T(\xi) \| \leqq M(\beta)e^{\beta\xi}$ we get for $\lambda > \beta$

$$\| [\lambda R(\lambda; A)]^k y \| \leqq \left(1 - \frac{\beta}{\lambda}\right)^{-k} M(\beta) \| y \|$$

and for $k > 2\beta\xi$

$$\left\| \left[\frac{k}{\xi} R\left(\frac{k}{\xi}; A\right)\right]^k y \right\| \leqq \left(1 - \frac{\beta\xi}{k}\right)^{-k} M(\beta) \| y \| < e^{2\beta\xi} M(\beta) \| y \| .$$

It follows that

$$\left\|\left[\frac{k}{\xi}R\left(\frac{k}{\xi};A\right)\right]^{k+1}x-\left[\frac{k}{\xi}R\left(\frac{k}{\xi};A\right)\right]^{k}x\right\|<e^{2\beta\xi}M(\beta)\left\|\frac{k}{\xi}R\left(\frac{k}{\xi};A\right)x-x\right\|$$

and this tends to 0 when $k \to \infty$. This completes the proof.

Formula (11.7.8) is the analog of (11.3.6) in the strong case. On the other hand, it would be very difficult to generalize (11.3.7) to the strong case. It seems likely that in general

$$\lim_{k\to\infty}\left[I+\frac{\xi}{k}A\right]^{k}x$$

does not exist even if x is restricted to belong to $\bigcap_n \mathfrak{D}[A^n]$.

11.8. Extensions. The preceding theory has been derived under the assumption that condition C_0 holds. Its validity is not restricted to this case, however. A minimal set of conditions is:

M. *For every* x

(i) $T(\xi)x$ *is measurable for* $\xi > 0$;

(ii) $\| T(\xi)x \|$ *is integrable in* $(0, 1)$;

(iii) $\lim_{\eta\to 0} \eta^{-1}\int_0^\eta T(\tau)x\,d\tau = x$.

The integral defining $R(\lambda; A)x$ has a sense if and only if the first two conditions hold. M(iii) implies that conclusion (1) under Theorem 11.6.1 holds and it is also necessary for the validity of the first part of this conclusion as is seen from the proof. M(iii) further implies that elements of the form $\int_\alpha^\beta T(\tau)x\,d\tau$ belong to $\mathfrak{D}[A]$ and that the latter is dense in $\mathfrak{X}$. Since $T(\xi)x$ is a continuous function of ξ for $x \in \mathfrak{D}[A]$, conclusion (2) under Theorem 11.6.1 still holds. Indeed, the proof of (2) now shows that $x^*(x) = 0$ for all $x \in \mathfrak{D}[A]$ and hence for all x. In the proof of (3) we use a preliminary integration by parts to show that $\int_0^\eta T(\tau)x\,d\tau \equiv \theta$ and hence that $x \equiv \theta$ by M(iii). Thus Theorem 11.6.1 holds under the new assumptions.

Similarly M(iii) suffices for the validity of (11.6.4) so that Theorems 11.6.2 and 11.6.3 are also true. Of the inversion theorems, 11.7.1, 11.7.3, and 11.7.4 are completely unaffected by the change in hypotheses. Theorems 11.7.2 and 11.7.5 have to be modified to the extent that the limits in question now hold almost everywhere.

11.9. The exponential formulas. It has been emphasized in several places in these Lectures that the semi-group operator $T(\xi)$ is an exponential function, $T(\xi) = \exp(\xi A)$, of its infinitesimal generator A. In the uniform case, the interpretation is straight forward

$$T(\xi) = \sum_{n=0}^{\infty}\frac{\xi^n}{n!}A^n x,$$

the generator being bounded, and the series converges in the uniform topology of $\mathfrak{E}(\mathfrak{X})$. In the strong case, A is an unbounded operator and the definition of

the exponential function is less direct. We collect here the nine exponential formulas proved so far which serve as justification of our use of the symbol $\exp(\xi A)\,x$ for $T(\xi)x$. They are formulas (9.3.10) = (9.3.15), (9.4.1) with $J = I$, and seven of the formulas of section 11.7. Unless otherwise stated, x is an arbitrary element of $\mathfrak{X}$. In general the limit relations hold uniformly with respect to ξ in any interval of the form $(\epsilon, 1/\epsilon)$; in the case of (E_1) the interval is $(\alpha, \alpha + 1/\epsilon)$, in ($E_2$) and ($E_5$) it is $(0, 1/\epsilon)$ instead.

$$(E_1) \qquad T(\xi)x = \lim_{\eta\to 0} \sum_{n=0}^{\infty} \frac{(\xi - \alpha)^n}{n!} \Delta_\eta^n T(\alpha)x, \qquad \alpha \geqq 0;$$

$$(E_2) \qquad T(\xi)x = \lim_{\eta\to 0} \exp[\xi A_\eta]\, x;$$

$$(E_3) \qquad T(\xi)x = \sum_{k=0}^{n-1} \frac{\xi^k}{k!} A^k x + \frac{1}{(n-1)!} \int_0^\xi (\xi - \tau)^{n-1} T(\tau) A^n x \, d\tau, \quad x \in \mathfrak{D}[A^n];$$

$$(E_4) \qquad \lim_{\xi\to 0} \xi^{-n} \left[T(\xi)x - \sum_{k=0}^{n-1} \frac{\xi^k}{k!} A^k x \right] = \frac{1}{n!} A^n x, \qquad \begin{cases} n = 1, 2, 3, \cdots, \\ x \in \bigcap_n \mathfrak{D}[A^n]; \end{cases}$$

$$(E_5) \qquad \int_0^\xi T(\tau)x\, d\tau = \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda\xi} R(\lambda; A)x \frac{d\lambda}{\lambda};$$

$$(E_6) \qquad T(\xi)x = \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda\xi} R(\lambda; A)x\, d\lambda, \qquad x \in \mathfrak{D}[A];$$

$$(E_7) \qquad T(\xi)x = \frac{1}{2\pi i} (C, 1)\text{-}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda\xi} R(\lambda; A)x\, d\lambda;$$

$$(E_8) \qquad T(\xi)x = \lim_{k\to\infty} \left[\frac{k}{\xi} R\left(\frac{k}{\xi}; A\right) \right]^k x;$$

$$(E_9) \qquad T(\xi)x = \lim_{k\to\infty} \left[I - \frac{\xi}{k} A \right]^{-k} x.$$

Other exponential formulas will be encountered later, in particular in connection with analytical semi-groups.

The formulas listed above separate into several distinct groups according to the device used in defining the exponential function. We shall encounter some of these devices in a more general setting in Chapter XV, and we shall postpone the analysis of the underlying principles until that chapter.

CHAPTER XII

GENERATION OF SEMI-GROUPS

12.1. Orientation. We come now to the converse problem of the theory: *What properties should an operator U possess in order that it be the infinitesimal generator of a continuous semi-group $\mathfrak{S} = \{T(\xi)\}$, $\xi > 0$, of linear bounded operators on a complex (B)-space to itself?*

Here the expected type of continuity must be specified. If we require that $T(\xi)$ tend uniformly to I when $\xi \to 0$, then the solution of the problem is trivial: *U must be a linear bounded operator and every such operator generates an analytical group.* There are, however, two interpretations of the continuity requirement which lead to worth while problems.

C_0. *$T(\xi)$ tends strongly but not uniformly to I when $\xi \to 0$.*

C_u. *C_0 holds but $T(\xi)$ is uniformly continuous for $\xi > 0$.*

We shall obtain sufficient conditions on the spectrum and the resolvent of U in order that the proposed problem have a solution satisfying C_0 or C_u. The method consists in showing that, under proper conditions, $R(\lambda; U)$ is the Laplace transform of a one-parameter operator $T(\xi)$ having the semi-group property. All three methods used below lead to semi-groups with rather special properties; the conditions imposed on the resolvent are somewhat arbitrary and only in one case can we assert that they are necessary for the desired result. Case C_u serves as a transition to analytical semi-groups discussed in the next chapter.

There are two paragraphs corresponding to the two problems indicated. There is no literature on this question.

1. Generation of a Strongly Continuous Semi-group

12.2. $R(\lambda;\ U)$ as a Laplace transform. In order to obtain an idea of what conditions should be imposed on the operator U, we review the available information concerning the resolvent of the infinitesimal generator of a semi-group $\mathfrak{S} = \{T(\xi)\}$ satisfying condition C_0.

This condition implies that $T(\xi)$ is strongly continuous for $\xi > 0$; its infinitesimal generator A is a closed unbounded operator whose domain $\mathfrak{D}[A]$ is dense in $\mathfrak{X}$ and the resolvent $R(\lambda; A)$ is the Laplace transform of $T(\xi)$

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda\xi}\, T(\xi)x\, d\xi, \qquad \lambda = \sigma + i\tau.$$

There are two properties of the resolvent which are of special interest to us: the half-plane in which $R(\lambda; A)$ is holomorphic and the behavior of $R(\lambda; A)$ for large values of $|\lambda|$. The first property is under our control; replacing $T(\xi)$ by $e^{-\alpha\xi}T(\xi)$ changes A into $A - \alpha I$ and shifts the spectrum to the left by the amount α. Without restricting the generality we may then assume that the spectrum of A is located in the left half-plane, $\sigma \leqq 0$. The behavior of $R(\lambda; A)$ for large values of $|\lambda|$ is regulated by the behavior of $T(\xi)$ for small values of ξ. Now condition C_0 implies that $\| T(\xi) \|$ is bounded for small ξ, but it does not imply that $\| T(\xi) \| \to 1$ when $\xi \to 0$. In a number of cases of interest to the applications, the latter condition holds, however, and even in the stronger form

$$\| T(\xi) \| \leqq 1 + \beta\xi, \qquad 0 \leqq \beta, 0 < \xi < 1. \tag{12.2.1}$$

As is easily verified, this condition implies that

$$\| R(\sigma + i\tau; A) \| \leqq \frac{1}{\sigma} + \frac{1}{\sigma^2}(\beta + \epsilon), \qquad \sigma > \sigma(\epsilon).$$

After this orientation we proceed to state

THEOREM 12.2.1. *Let U be a closed, linear, unbounded operator on $\mathfrak{X}$ to itself whose domain is dense in $\mathfrak{X}$. Let the spectrum of U be located in $\mathfrak{R}(\lambda) = \sigma \leqq 0$ and suppose that*

$$\| R(\sigma + i\tau; U) \| \leqq \frac{1}{\sigma} + \frac{\beta}{\sigma^2}, \qquad \sigma > 0, \tag{12.2.2}$$

where β is a fixed constant, $\beta \geqq 0$. Then U is the infinitesimal generator of a semi-group $\mathfrak{S} = \{T(\xi)\}$, $\xi > 0$; $T(\xi)$ satisfies condition C_0 and $\| T(\xi) \| \leqq e^{\beta\xi}$ for all $\xi > 0$. Further

$$T(\xi)x = \lim_{k\to\infty} \left\{ \frac{k}{\xi} R\left(\frac{k}{\xi}; U\right) \right\}^k x, \qquad x \in \mathfrak{X}.$$

The proof of this theorem will be given in a number of steps which we formulate as lemmas and intermediary theorems. We start by observing that for each fixed λ with $\sigma > 0$ the whole space $\mathfrak{X}$ is the domain of $R(\lambda; U)$ and the range of $\lambda I - U$, while $\mathfrak{D}[U]$ is the domain of $\lambda I - U$ and the range of $R(\lambda; U)$.

LEMMA 12.2.1. *$\mathfrak{D}[U^n]$ is dense in $\mathfrak{X}$ for all n.*

PROOF. We need only the case $n = 2$ which is proved as follows. For any fixed λ with $\sigma > 0$ the linear bounded operator $R(\lambda; U)$ maps $\mathfrak{X}$ onto $\mathfrak{D}[U]$ which is dense in $\mathfrak{X}$. It will then map the subset $\mathfrak{D}[U]$ onto a subset $\mathfrak{D}_2$ which is dense in $\mathfrak{D}[U]$ and hence in $\mathfrak{X}$. Every element of $\mathfrak{D}_2$, being of the form $R^2(\lambda; U)y$, belongs to $\mathfrak{D}[U^2]$ so the latter set is dense in $\mathfrak{X}$.

The next three theorems serve to prove that $R(\lambda; U)$ is a Laplace transform.

THEOREM 12.2.2. *If* $\gamma > 0, \xi > 0$, *and* $x \in \mathfrak{D}[U^2]$, *then*

$$T(\xi)x = \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{\lambda\xi} R(\lambda; U)x \, d\lambda \tag{12.2.3}$$

defines a linear bounded transformation on $\mathfrak{D}[U^2]$ *to* $\mathfrak{X}$. $T(\xi)x$ *is continuous for* $\xi \geqq 0$ *and*

$$R(\lambda; U)x = \int_0^\infty e^{-\lambda\xi} T(\xi)x \, d\xi, \qquad x \in \mathfrak{D}[U^2]. \tag{12.2.4}$$

PROOF. For $x \in \mathfrak{D}[U^2]$ we have

$$R(\lambda; U)x = \frac{1}{\lambda} x + \frac{1}{\lambda^2} Ux + \frac{1}{\lambda^2} R(\lambda; U)U^2 x. \tag{12.2.5}$$

Substitution in (12.2.3) gives

$$T(\xi)x = x + \xi Ux + \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda\xi} R(\lambda; U)U^2 x \frac{d\lambda}{\lambda^2}, \tag{12.2.6}$$

where the integral converges absolutely for $\xi \geqq 0$. It also converges uniformly with respect to ξ in every finite interval $[0, \omega]$ whence it follows that $T(\xi)x$ is a continuous function of ξ. The norm of the integral is dominated by

$$\frac{1}{2\gamma^2}\left(1 + \frac{\beta}{\gamma}\right) e^{\gamma\xi} \| U^2 x \|,$$

as is shown by (12.2.2). Here we may take $\gamma = 2/\xi$ when $\xi > 0$, obtaining

$$\| T(\xi)x \| \leqq \| x \| + \xi \| Ux \| + \xi^2(1 + \tfrac{1}{2}\beta\xi) \| U^2 x \|.$$

Thus if we renorm $\mathfrak{D}[U^2]$, taking

$$\| x \|_1 = \| x \| + \| Ux \| + \| U^2 x \|$$

as the new norm, we see that $T(\xi)$ defines a bounded linear transformation on $\mathfrak{D}[U^2]$ to $\mathfrak{X}$. Since $T(\xi)$ is continuous and bounded in the manner indicated, the integral in (12.2.4) is absolutely convergent for $\sigma > 0$. For $\sigma > \gamma$ we may substitute the expression for $T(\xi)x$ in the integral, obtaining

$$\int_0^\infty e^{-\lambda\xi} T(\xi)x \, d\xi = \frac{1}{\lambda} x + \frac{1}{\lambda^2} Ux + \frac{1}{2\pi i} \int_0^\infty e^{-\lambda\xi} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\mu\xi} R(\mu; U)U^2 x \frac{d\mu}{\mu^2} \, d\xi.$$

The double integral being absolutely convergent, we may interchange the order of integration, obtaining

$$\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{R(\mu; U)U^2 x}{\lambda - \mu} \frac{d\mu}{\mu^2} = \frac{1}{\lambda^2} R(\lambda; U)U^2 x.$$

Here the value of the integral may be found by the calculus of residues in the classical manner. Comparing the result with (12.2.5), we see that the proof is complete.

So far we have not used condition (12.2.2) in any essential manner. The same results could be proved under less restrictive assumptions on the resolvent. The condition will be used fully in the next theorem, however. Putting

$$T_k(\xi)x = \left[\frac{k}{\xi} R\left(\frac{k}{\xi}; U\right)\right]^k x \tag{12.2.7}$$

we shall prove

THEOREM 12.2.3. *For all $\xi > 0$ and all $x \in \mathfrak{X}$*

$$\lim_{k\to\infty} T_k(\xi)x \tag{12.2.8}$$

exists and coincides with $T(\xi)x$ when $x \in \mathfrak{D}[U^2]$.

PROOF. Since $R(\lambda; U)x$ is the Laplace transform of $T(\xi)x$ when $x \in \mathfrak{D}[U^2]$, Theorem 11.7.5 applies and shows that the limit in (12.2.8) exists and equals $T(\xi)x$ for $x \in \mathfrak{D}[U^2]$ and $\xi > 0$. It should be noted that in applying Theorem 11.7.5 we need to know only that $R(\lambda; U)x$ is the Laplace transform of the continuous function $T(\xi)x$ and that $R(\lambda; U)$ is a resolvent. The result can also be obtained directly from Theorem 10.3.4.

Next we note that condition (12.2.2) gives

$$\| T_k(\xi) \| \leqq \left[1 + \frac{\beta\xi}{k}\right]^k \leqq e^{\beta\xi} \tag{12.2.9}$$

for all k and $\xi > 0$. Further $\mathfrak{D}[U^2]$ is dense in $\mathfrak{X}$; hence by Theorem 2.12.1 (the Banach-Steinhaus theorem) the limit in (12.2.8) exists for all $x \in \mathfrak{X}$. This completes the proof.

We now define $T(\xi)x$ for all x as the limit in (12.2.8) and note that

$$\| T(\xi) \| \leqq e^{\beta\xi}, \qquad \xi > 0. \tag{12.2.10}$$

THEOREM 12.2.4. *$T(\xi)$ is strongly measurable and* (12.2.4) *holds for all $x \in \mathfrak{X}$, $\sigma > \beta$.*

PROOF. $T(\xi)x$ is the limit of $T_k(\xi)x$, that is, of a sequence of continuous functions of ξ; it follows that $T(\xi)x$ is a measurable function of ξ. Hence the integral in (12.2.4) exists for all x and defines a linear bounded transformation on $\mathfrak{X}$ to itself for each fixed λ with $\sigma > \beta$ since (12.2.10) holds. But this operator coincides with the bounded linear operator $R(\lambda; U)$ on the dense set $\mathfrak{D}[U^2]$ so the operators must be identical.

12.3. The semi-group property. In order to complete the proof of Theorem 12.2.1 we have to show that the operator $T(\xi)$ defined by (12.2.8) is strongly continuous and has the semi-group property. Once this has been accomplished, formula (12.2.4) shows that U is the infinitesimal generator of $T(\xi)$.

THEOREM 12.3.1. *$T(\xi)$ is strongly continuous for $\xi > 0$.*

PROOF. This follows from the fact that (i) $T(\xi)x$ is continuous when $x \in \mathfrak{D}[U^2]$ which is dense in $\mathfrak{X}$, and (ii) formula (12.2.10) holds. The details are left to the reader.

THEOREM 12.3.2. *$T(\xi)$ tends strongly to I when $\xi \to 0$.*

PROOF. This follows from the Banach-Steinhaus theorem since $\lim_{\xi\to 0} T(\xi)x = x$ for x in the dense set $\mathfrak{D}[U^2]$ and (12.2.10) holds.

THEOREM 12.3.3. *For $\xi > 0$, $x \in \mathfrak{X}$, and $n = 0, 1, 2, \cdots$*

$$\xi^{n+1} T(\xi)x = (n + 1) \int_0^\xi \eta^n T(\xi - \eta)T(\eta)x \, d\eta. \tag{12.3.1}$$

PROOF. This relation is a consequence of the identity

$$R^{(n+1)}(\lambda; U)x = -(n + 1)R(\lambda; U)[R^{(n)}(\lambda; U)x] \tag{12.3.2}$$

which is obtained from formula (5.8.3). Here we substitute the representations by Laplace integrals, obtaining

$$\begin{aligned}\int_0^\infty e^{-\lambda\xi} \xi^{n+1} T(\xi)x \, d\xi &= (n + 1) \int_0^\infty e^{-\lambda\phi} T(\phi) \left\{ \int_0^\infty e^{-\lambda\psi} \psi^n T(\psi)x \, d\psi \right\} d\phi \\ &= (n + 1) \int_0^\infty e^{-\lambda\xi} \left\{ \int_0^\xi \eta^n T(\xi - \eta)T(\eta)x \, d\eta \right\} d\xi.\end{aligned}$$

The last step is proved by the same type of argument as used in the proof of Theorem 10.2.4. It is an easy matter to prove that

$$\int_0^\xi \eta^n T(\xi - \eta)T(\eta)x \, d\eta$$

is a continuous function of ξ since $T(\xi)x$ has this property. By the uniqueness theorem for Laplace integrals, Theorem 10.2.3, we conclude that (12.3.1) holds.

We can now prove the final step:

THEOREM 12.3.4. *$T(\xi)$ has the semi-group property and U is the infinitesimal generator of $T(\xi)$.*

PROOF. If x^* is any bounded linear functional on $\mathfrak{X}$, then formula (12.3.1) implies that

$$\int_0^\xi \eta^n x^*[T(\xi)x - T(\xi - \eta)T(\eta)x] \, d\eta = 0, \qquad n = 0, 1, 2, \cdots.$$

But here $x^*[\cdot]$ is a continuous numerical function of η in the interval $(0, \xi)$ and if such a function is orthogonal to all powers of η, then it must be identically

zero. From $x^*[\cdot] = 0$ for all x^*, we conclude that

$$T(\xi)x = T(\xi - \eta)T(\eta)x$$

for all ξ, η, and x, $0 < \eta < \xi$, $x \in \mathfrak{X}$. This proves that $\mathfrak{S} = \{T(\xi)\}$ is a semi-group of linear bounded operators on $\mathfrak{X}$ to itself. That U is the infinitesimal generator of $\mathfrak{S}$ follows from (12.2.4). We have already seen that $T(\xi)$ tends strongly to I when $\xi \to 0$; it cannot tend uniformly to I since this would make U bounded against our assumption. This completes the proof of Theorem 12.2.1.

It should be noted that the conditions of Theorem 12.2.1 are not sufficient for property C_u. A counter example is provided, for instance, by the semi-group of right translations on $C[0, \infty]$ so that $T(\xi)[x(t)] = x(t + \xi)$. Here the conditions of Theorem 12.2.1 are satisfied with $\beta = 0$ and $T(\xi)$ satisfies C_0 but not C_u. Cf. section 16.2 for a discussion of this semi-group.

When $\beta = 0$ Theorem 12.2.1 is the best of its kind in the sense that the stated conditions are necessary as well as sufficient for the validity of the conclusions. We lose the necessity when $\beta > 0$; however, if C_0 is satisfied and $\| T(\xi) \| \leqq e^{\beta\xi}$ for $\xi > 0$, then a condition of form (12.2.2) with β replaced by $\beta + \epsilon$ holds for large values of σ. As a matter of fact, we need (12.2.2) only for large values of σ provided we know in addition that $\| R(\sigma + i\tau; U) \|$ is bounded for $\sigma \geqq \delta > 0$. Thus the sufficient conditions are very nearly necessary also for $\beta > 0$.

Theorem 12.2.1 can be considered only as a partial solution of the problem proposed in section 12.1 inasmuch as the corresponding semi-group operators have norms with very special properties for small values of ξ. Cf. formula (12.2.1). Such a behavior of the norm is by no means implied by condition C_0 but follows from the restrictive assumption (12.2.2). It should be possible to carry through the argument assuming merely that $\sigma \| R(\sigma + i\tau; U) \|$ is bounded for $\sigma > 0$, but it does not seem possible to base the proof of Theorem 12.2.3 on this assumption.

2. Generation of Semi-groups Uniformly Continuous for $\xi > 0$

12.4. Properly dominated resolvents. We shall investigate two different sets of assumptions on the operator U which lead to semi-groups satisfying condition C_u. It is of course necessary that the resolvent $R(\lambda; U)$ be holomorphic in a half-plane which we may take as $\sigma > 0$. In addition we must impose conditions on the resolvent which are more stringent than those of Theorem 12.2.1. We can either impose stronger restrictions on the rate of growth of $R(\lambda; U)$ in terms of the distance of λ from the spectrum or else keep the distant part of the spectrum away from the imaginary axis. Both devices can be manipulated so as to give the desired result. The two classes of semi-groups which may be constructed in this manner overlap but are distinct; both classes contain non-analytic elements. In the first case we obtain

THEOREM 12.4.1. *Let U be a closed, linear, unbounded operator on $\mathfrak{X}$ to itself whose domain is dense in $\mathfrak{X}$. Let the spectrum of U be located in $\sigma \leqq 0$. Suppose further that for $\sigma > 0$*

$$\sigma \int_{-\infty}^{\infty} \| R(\sigma + i\tau; U) \|^2 \, d\tau \leqq B. \tag{12.4.1}$$

Then U is the infinitesimal generator of a semi-group $\mathfrak{S} = \{T(\xi)\}$ satisfying condition C_u. $T(\xi)$ need not be differentiable, much less analytic. For $\xi > 0, \gamma > 0$ we have

$$\xi T(\xi) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda \xi} R^2(\lambda; U) \, d\lambda. \tag{12.4.2}$$

REMARK. Condition (12.4.1) is satisfied if, for instance, $|\lambda| \, \| R(\lambda; U) \|$ is bounded for $\Re(\lambda) > 0$, or more generally, if for $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$ we have $\rho \, \| R(\rho e^{i\phi}; U) \| \leqq M(\phi)$ where $M(\phi) \in L_2(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

PROOF. Condition (12.4.1) shows that $R(\lambda; U) \in H_2[\epsilon; \mathfrak{E}(\mathfrak{X})]$ for every $\epsilon > 0$. By Theorem 10.4.2

$$\lim_{|\tau| \to \infty} \| R(\sigma + i\tau; U) \| = 0, \qquad \sigma > 0. \tag{12.4.3}$$

It follows from (12.4.1) that the integral in (12.4.2) exists for every real ξ and by (12.4.3) the value of the integral is independent of γ as long as $\gamma > 0$. Moreover the integral converges uniformly with respect to ξ in every finite interval so that its value which for $\xi > 0$ is denoted by $\xi T(\xi)$ is a continuous function of ξ. For $\xi < 0$ the value of the integral is θ, as we see by letting $\gamma \to \infty$, and by continuity the same is true for $\xi = 0$. Adding the integrals for ξ and $-\xi$ we get the alternate representation

$$T(\xi) = \frac{1}{\xi} \frac{1}{\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \sinh(\lambda \xi) \, R^2(\lambda; U) \, d\lambda, \qquad \xi > 0. \tag{12.4.4}$$

Using (12.4.1) and choosing $\gamma = 1/\xi$ we obtain the estimate

$$\| T(\xi) \| \leqq \frac{e}{2\pi} B. \tag{12.4.5}$$

Thus $T(\xi)$ is an element of $\mathfrak{E}(\mathfrak{X})$ and $\| T(\xi) \|$ is uniformly bounded.

We observe next that if

$$T(\xi; \omega) = \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda \xi} R(\lambda; U) \, d\lambda \tag{12.4.6}$$

then

$$\lim_{\omega \to \infty} T(\xi; \omega) = T(\xi). \tag{12.4.7}$$

This is shown by an integration by parts in (12.4.6) using (12.4.3) and $R'(\lambda; U) = -R^2(\lambda; U)$. Thus formula (12.4.2) is simply another variant of the complex inversion formula for Laplace integrals.

We shall now verify that $T(\xi)$ has the semi-group property. We form

$$T(\xi_1)T(\xi_2) = \frac{1}{\xi_1\xi_2}\frac{1}{(2\pi i)^2}\int\int e^{\lambda\xi_1+\mu\xi_2}R^2(\lambda; U)R^2(\mu; U)\,d\lambda\,d\mu,$$

taking the integrals along $\Re(\lambda) = \gamma_1$, $\Re(\mu) = \gamma_2$ where $0 < \gamma_1 < \gamma_2$. The double integral is clearly absolutely convergent. Repeated use of the first resolvent equation gives

$$R^2(\lambda; U)R^2(\mu; U) = \frac{R^2(\lambda; U)}{(\lambda-\mu)^2} + 2\,\frac{R(\lambda; U) - R(\mu; U)}{(\lambda-\mu)^3} + \frac{R^2(\mu; U)}{(\lambda-\mu)^2}.$$

Thus $T(\xi_1)T(\xi_2)$ becomes the sum of three absolutely convergent double integrals. In the first one we integrate first with respect to μ and then with respect to λ, obtaining

$$\frac{1}{\xi_1\xi_2}\frac{1}{2\pi i}\int e^{\lambda\xi_1}R^2(\lambda; U)\left\{\frac{1}{2\pi i}\int e^{\mu\xi_2}\frac{d\mu}{(\lambda-\mu)^2}\right\}d\lambda$$

$$= \frac{1}{\xi_1}\frac{1}{2\pi i}\int e^{\lambda(\xi_1+\xi_2)}R^2(\lambda; U)\,d\lambda = \frac{\xi_1+\xi_2}{\xi_1}\,T(\xi_1+\xi_2).$$

In the third integral integration with respect to λ gives zero so this term drops out. The second integral is the limit when $\omega \to \infty$ of

$$\frac{1}{\xi_1\xi_2}\frac{1}{(2\pi i)^2}\left\{\int_{\gamma_1-i\omega}^{\gamma_1+i\omega}\int_{\gamma_2-i\infty}^{\gamma_2+i\infty} e^{\lambda\xi_1+\mu\xi_2}\frac{R(\lambda; U)}{(\lambda-\mu)^3}\,d\mu\,d\lambda\right.$$

$$\left. - \int_{\gamma_2-i\omega}^{\gamma_2+i\omega}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty} e^{\lambda\xi_1+\mu\xi_2}\frac{R(\mu; U)}{(\lambda-\mu)^3}\,d\lambda\,d\mu\right\}.$$

Carrying out the integrations over the infinite intervals, one gets

$$-\frac{\xi_2}{\xi_1}\frac{1}{2\pi i}\int_{\gamma_1-i\omega}^{\gamma_1+i\omega} e^{\lambda(\xi_1+\xi_2)}R(\lambda; U)\,d\lambda = -\frac{\xi_2}{\xi_1}\,T(\xi_1+\xi_2\,;\omega)$$

and this expression tends to $-\xi_1^{-1}\xi_2T(\xi_1+\xi_2)$ when $\omega \to \infty$ by (12.4.7). Combining the values of the three integrals, we get the required relation

$$T(\xi_1+\xi_2) = T(\xi_1)T(\xi_2).$$

Another lengthy computation is required to verify the formula

$$R(\lambda; U) = \int_0^\infty e^{-\lambda\xi}T(\xi)\,d\xi, \qquad \Re(\lambda) > 0. \tag{12.4.8}$$

The integral exists since $T(\xi)$ is continuous and satisfies (12.4.5). In order to compute the value of the integral we assume $\Re(\lambda) > \gamma$ and substitute the expression in (12.4.4) for $T(\xi)$. By a permissible change of the order of integration we get

$$\frac{1}{\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} R^2(\mu; U)\,d\mu\int_0^\infty e^{-\lambda\xi}\sinh(\mu\xi)\,\frac{d\xi}{\xi}.$$

Here the integral with respect to ξ can be evaluated by differentiating with respect to μ, observing that the integral should vanish for $\mu = 0$. The value is found to be

$$\tfrac{1}{2} \log \frac{\lambda + \mu}{\lambda - \mu},$$

where the logarithm is supposed to be real when λ and μ are real, $\mu < \lambda$. The iterated integral consequently equals

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} R^2(\mu; U) \log \frac{\lambda + \mu}{\lambda - \mu}\, d\mu.$$

An integration by parts reduces this to

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} R(\mu; U) \left[\frac{1}{\lambda - \mu} + \frac{1}{\lambda + \mu}\right] d\mu = R(\lambda; U).$$

Here we have used the fact that the integrated part vanishes in the limits by (12.4.3) since the logarithmic factor stays bounded. Further, $R(\lambda; U)$ as an element of $H_2[\epsilon; \mathfrak{E}(\mathfrak{X})]$ is represented by its proper Cauchy integral while the integral involving the kernel $(\lambda + \mu)^{-1}$ equals θ as is seen by letting $\gamma \to \infty$. This completes the proof of formula (12.4.8).

The final steps are easy to take. We have to verify that $T(\xi)$ satisfies condition C_0 and that U really is the infinitesimal generator of $T(\xi)$. As in the proof of Theorem 12.2.2 we have

$$T(\xi)x = x + \xi Ux + O(\xi^2), \qquad x \in \mathfrak{D}[U^2].$$

From this we see first that $\lim_{\xi \to 0} T(\xi)x = x$ for all x in the dense set $\mathfrak{D}[U^2]$; since $\| T(\xi) \|$ is uniformly bounded for $\xi > 0$, the Banach-Steinhaus theorem applies and shows that $\lim_{\xi \to 0} T(\xi)x = x$ for all x in $\mathfrak{X}$. Thus condition C_u holds. The formula also shows that U is the infinitesimal generator of $T(\xi)$.

An example of a trigonometric semi-group on $L_2(-\pi, \pi)$ which satisfies the conditions of Theorem 12.4.1 is given by

$$T(\xi)[x(t)] = \sum_{-\infty}^{\infty} \exp\left[-\xi(n^2 + i \operatorname{sgn}(n)\, e^{n^4})\right] x_n e^{nit}, \tag{12.4.9}$$

where x_n is the nth Fourier coefficient of $x(t) \in L_2(-\pi, \pi)$. A simple consideration shows that the infinitesimal generator A as well as $AT(\xi) = T'(\xi)$ are unbounded operators on $L_2(-\pi, \pi)$. From the second fact it follows that $T(\xi)$ does not have a derivative in L_2 even in the weak sense. This completes the proof of the theorem.

12.5. Domains of type C_u. For the second method we impose sharper restrictions on the spectrum of U; the resolvent set shall contain essentially more than a half-plane without being allowed to contain a sector of opening greater than π. The exact limitation is given in

DEFINITION 12.5.1. *A simply-connected domain Δ of the complex λ-plane $(\lambda = \sigma + i\tau)$ is said to be of type C_u if*

(i) *it contains the positive real axis;*

(ii) *the boundary of Δ is a curve $\Gamma: \sigma = -\psi(\tau)$, where $\psi(\tau)$ is a non-negative continuous function of τ which never decreases when $|\tau|$ increases;*

(iii) *$\psi(0) = 0$, $\psi(\tau) \to \infty$ when $|\tau| \to \infty$ but $\psi(\tau)/\tau \to 0$;*

(iv) *$\psi'(\tau)$ is equivalent to a bounded function and $\psi'(\tau) \to 0$ when $|\tau| \to \infty$;*

(v) *$\int_{-\infty}^{\infty} \exp[-\xi\psi(\tau)]\, d\tau$ exists for $\xi > 0$.*

Here condition (v) is satisfied if, for instance, $\psi(\tau)[\log|\tau|]^{-1} \to \infty$ when $|\tau| \to \infty$. We shall now prove

THEOREM 12.5.1. *Let U be a closed, linear, unbounded operator on $\mathfrak{X}$ to itself whose domain is dense in $\mathfrak{X}$. Let the resolvent set of U contain a domain Δ of type C_u. Let*

$$\delta(\lambda)\, \| R(\lambda; U) \| \leqq M \tag{12.5.1}$$

for all λ in Δ where $\delta(\lambda)$ is the distance from λ to the boundary Γ of Δ. Then U is the infinitesimal generator of a semi-group $\mathfrak{S} = \{T(\xi)\}$, $\xi > 0$, such that $T(\xi)$ satisfies condition C_u. $T(\xi)$ has derivatives of all orders but need not be an analytic function of ξ. $T(\xi)$ is given by formula (12.5.2) below.

PROOF. Let $T(\xi; \omega)$ be defined by (12.4.6). Since (12.5.1) implies $\sigma \| R(\sigma + i\tau; U) \| \leqq M$ for $\sigma > 0$, the argument used in section 12.2 shows that $\lim_{\omega\to\infty} T(\xi; \omega)x \equiv T(\xi)x$ exists when $x \in \mathfrak{D}[U^2]$. If we can show the existence of $\lim_{\omega\to\infty} T(\xi; \omega)$ in the uniform topology, then the limit must be an extension of $T(\xi)$ from $\mathfrak{D}[U^2]$ to all of $\mathfrak{X}$. Furthermore, the argument used in section 12.3 shows that $T(\xi)$ is a semi-group operator generated by U and that $T(\xi)$ satisfies C_0. To accomplish the proof we have then merely to show the existence of $\lim_{\omega\to\infty} T(\xi; \omega) \equiv T(\xi)$ in the uniform sense and to prove that $T(\xi)$ has uniform derivatives of all orders. Condition C_u then follows.

For this purpose we consider the integral

$$\frac{1}{2\pi i} \int_{\Gamma_1(\omega)} e^{\lambda\xi} R(\lambda; U)\, d\lambda = \Theta$$

and let $\omega \to \infty$. Here $\Gamma_1(\omega)$ is the closed contour $ABCDA$ where

$$A = \gamma + i\omega,\ B = -\tfrac{1}{2}\psi(\omega) + i\omega,\ C = -\tfrac{1}{2}\psi(-\omega) - i\omega,\ D = \gamma - i\omega.$$

BC is an arc of the curve $\Gamma_1: \sigma = -\frac{1}{2}\psi(\tau)$, a small arc containing the origin being replaced by a circular arc in Δ; γ exceeds the radius of this circle, and the other portions of $\Gamma_1(\omega)$ are straight line segments. The integral along DA equals $T(\xi; \omega)$.

The distance $\delta(\lambda)$ from a point $\lambda = -\frac{1}{2}\psi(\tau) + i\tau$ on Γ_1 to Γ exceeds $C\psi(\tau)$ by virtue of (iv), C being a fixed constant. It follows that the horizontal portions of the path give contributions to the integral the norms of which

do not exceed a constant multiple of

$$\frac{1}{\psi(\omega)} \int_{-\infty}^{\gamma} e^{\sigma\xi}\, d\sigma = \frac{e^{\gamma\xi}}{\xi\psi(\omega)}$$

and this tends to zero when $\omega \to \infty$. The integral along CB tends to

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp\, [\xi(-\tfrac{1}{2}\psi(\tau) + i\tau)]\, R[-\tfrac{1}{2}\psi(\tau) + i\tau; U]\, [i - \tfrac{1}{2}\psi'(\tau)]\, d\tau,$$

with an obvious redefinition of $\psi(\tau)$ for small values of τ on the circular arc. This integral is absolutely convergent by (v) since its norm is dominated by a constant multiple of

$$\int_{-\infty}^{\infty} \exp\left[-\frac{\xi}{2}\psi(\tau)\right] [\psi(\tau)]^{-1}\, d\tau.$$

It follows that $T(\xi; \omega)$ has a limit in the uniform topology and $T(\xi)$ may be represented by the absolutely convergent integral

$$T(\xi) = \frac{1}{2\pi i} \int_{\Gamma_1} e^{\lambda\xi} R(\lambda; U)\, d\lambda. \tag{12.5.2}$$

This integral converges uniformly with respect to ξ in every interval $(\epsilon, 1/\epsilon)$, $\epsilon > 0$, whence it follows that $T(\xi)$ is continuous also in the uniform topology for $\xi > 0$. Moreover, we may differentiate with respect to ξ under the sign of integration as often as we please so that

$$T^{(n)}(\xi) = \frac{1}{2\pi i} \int_{\Gamma_1} e^{\lambda\xi} \lambda^n R(\lambda; U)\, d\lambda.$$

To see that this integral is also absolutely convergent we note that its norm is dominated by a constant multiple of

$$\int_{-\infty}^{\infty} \exp\left[-\frac{\xi}{2}\psi(\tau)\right] |\, \tau\, |^n\, d\tau \leqq \left\{M\left(\frac{\xi}{2n+2}\right)\right\}^n \int_{-\infty}^{\infty} \exp\left[-\frac{\xi}{2n+2}\psi(\tau)\right] d\tau$$

where $M(\eta) = \max_\tau |\, \tau\, | \exp\, [-\eta\psi(\tau)]$ is finite since

$$\int_{\alpha/2}^{\alpha} \exp\, [-\eta\psi(\tau)]\, d\tau > \frac{\alpha}{2} \exp\, [-\eta\psi(\alpha)]$$

and the left side tends to zero when $\alpha \to \infty$. A similar estimate holds for negative values of α.

Finally we have to exhibit an operator U satisfying the conditions of Theorem 12.5.1 such that the corresponding semi-group operator is non-analytic. We take $\mathfrak{X} = L_2(-\pi, \pi)$ with the usual metric and define $T(\xi)$ by

$$T(\xi)[x(t)] = \sum_{-\infty}^{\infty} \exp\, [-\xi(|\, n\, | + i \operatorname{sgn}(n)\, n^2)]\, x_n e^{nit},$$

where the x_n's are the Fourier coefficients of $x(t)$. The generator of this semi-group is the unbounded transformation A:

$$A[x(t)] \sim -\sum_{-\infty}^{\infty} (|n| + i \operatorname{sgn}(n)\, n^2) x_n e^{nit} \tag{12.5.3}$$

with the resolvent

$$R(\lambda; A)[x(t)] = \sum_{-\infty}^{\infty} \frac{1}{\lambda + |n| + i \operatorname{sgn}(n) \cdot n^2} x_n e^{nit}.$$

From these expressions we read off that the spectrum of A consists of the points $\{-[|n| + i \operatorname{sgn}(n) n^2]\}$, $n = 0, \pm 1, \pm 2, \cdots$, and $\| R(\lambda; A) \|$ equals the reciprocal of the distance from λ to the nearest point of the spectrum. The resolvent set contains the domain $\sigma > -|\tau|^{\frac{1}{2}}$ which is clearly of type C_u. Thus A satisfies the conditions of Theorem 12.5.1. The terms with positive subscripts in the series for $T(\xi)[x(t)]$ define a function holomorphic in the lower half-plane, those with negative subscripts one which is holomorphic in the upper, and both functions have the real axis as natural boundary. Thus $T(\xi)$ cannot be analytic. This completes the proof of Theorem 12.5.1.

It should be observed, however, that the problems discussed in the present chapter call for further investigations with a view of obtaining necessary and sufficient conditions.

CHAPTER XIII

ANALYTICAL SEMI-GROUPS

13.1. Orientation. The semi-groups $\mathfrak{S} = \{T(\xi)\}$ which have been studied in the preceding chapters have been defined for real positive values of the parameter. The question of extending the parameter manifold and of finding the *maximal domain of definition* of $\mathfrak{S}$ is a very natural one.

The question is not well put, however, and has little sense unless we make clear what kind of extension is wanted. It is always possible to embed the given semi-group $\mathfrak{S}$ into a semi-group $\mathfrak{S}_0 = \{W(\zeta)\}$ defined for $\Re(\zeta) > 0$ such that $W(\xi) = T(\xi)$, $\xi > 0$. We may define, for instance,

$$W(\xi + i\eta) = T(\xi)e^{\beta\eta}$$

where β is an arbitrary complex number. Thus an extension of $T(\xi)$ from the positive real axis to the right half-plane preserving the semi-group property is always possible and is never unique.

On the other hand, we may raise the question if the given semi-group operator $T(\xi)$ admits of an *analytic extension* to a part of the complex plane and what properties such an extension may possess. Such an extension ordinarily does not exist; when it exists, it is unique and the extension has the semi-group property wherever it exists. Here it makes sense to ask what is the domain of existence of $T(\zeta)$ and we may refer to this domain as the maximal domain of definition of $\mathfrak{S}$.

It turns out that if $T(\xi)$ is analytic on an arbitrarily small interval $(0, \delta)$ of the real axis, having the origin as a limit point, then the domain of existence of $T(\zeta)$ is an angular semi-module in the sense of Chapter VII.

We shall determine various conditions under which $T(\xi)$ will admit of an analytic continuation to an angular semi-module Σ_2. For the case in which Σ_2 is a sector, the conditions are necessary and sufficient in order that the extension shall exist and have a prescribed rate of growth along the rays of the sector. The discussion is analogous to that of Chapters XI and XII. There are some new features, however. The relations between the rate of growth of $T(\zeta)$ in the sector as measured by the "indicator" and the position of the singularities of the resolvent $R(\lambda; A)$, that is, the spectrum of A, are of considerable interest. They are completely analogous to the corresponding relations in the numerically-valued case between the rate of growth of a determining function, analytic and of exponential type in a sector, and the position of the singularities of its Laplace transform. These are well known, at least in the case in which the sector is the whole plane (see G. Pólya [2]). If $T(\zeta)$ is holomorphic and of exponential type in a half-plane we can also use representations by binomial series and Laplace contour integrals to great advantage. Using

the latter device, we show that it is possible, under mild restrictions, to embed a given transformation in a semi-group which is analytic in a half-plane.

There are three paragraphs: *Domains of Analyticity*, *The Structure of Analytical Semi-Groups*, and *Semi-Groups and Interpolation Series*.

References. Carlson [2], Hille [5, 6, 7], Nörlund [2], Phragmén and Lindelöf [1], Pólya [2].

1. Domains of Analyticity

13.2. Analytic extension of semi-groups. Our main concern is to show that the semi-group property is preserved under analytic continuation and that the domain of analyticity of a semi-group operator is a semi-module. The following theorem is basic.

THEOREM 13.2.1. *Let Z be a convex domain in the complex ζ-plane, $(Z)_a$ its additive resultant in the sense of Definition* 7.5.2. *Let $\{\zeta_n\}$ be a countable point set in Z such that* (i) *$\{\zeta_n\}$ has a limit point ζ_0 in Z,* (ii) *$2\zeta_0 \in Z$, and* (iii) *$\zeta_j + \zeta_k \in Z$ for $j, k = 1, 2, 3, \cdots$. Let $\mathfrak{X}$ be a complex (B)-space, $\mathfrak{E}(\mathfrak{X})$ its (B)-algebra of endomorphisms. Let $W(\zeta)$ be a function on complex numbers to $\mathfrak{E}(\mathfrak{X})$ which is defined and holomorphic in Z. Let*

$$W(\zeta_j + \zeta_k) = W(\zeta_j)W(\zeta_k) \tag{13.2.1}$$

for all j and k. Then $(Z)_a$ is a simply-connected domain, $W(\zeta)$ may be continued analytically all over $(Z)_a$, and for all ζ' and ζ'' in $(Z)_a$

$$W(\zeta' + \zeta'') = W(\zeta')W(\zeta''). \tag{13.2.2}$$

PROOF. We recall that $(Z)_a$ is an open set since Z is open. If Z_n denotes the image of Z under the transformation $\zeta' = n\zeta$ where n is any positive integer, then each set Z_n is a convex domain and $(Z)_a = \bigcup_n Z_n$. By assumption (ii), $Z_1 \cap Z_2 \neq \emptyset$ so that Lemma 7.5.1 shows that $(Z)_a$ is connected and consequently simply-connected by Theorem 7.5.2. The analytic continuation requires a more elaborate argument.

Since $2\zeta_0 \in Z$, we may assume, without loss of generality, that $\zeta_0 + \zeta_j \in Z$ for all j. $W(\zeta)$ being continuous in Z, it follows that (13.2.1) holds when j or k or both are zero.

We now consider the two functions $W(\zeta)W(\alpha)$ and $W(\alpha)W(\zeta)$. They are well defined for ζ and α in Z and are holomorphic in each of the variables separately. Further they are equal when $\zeta = \zeta_j$, $\alpha = \zeta_k$, $j, k = 0, 1, 2, \cdots$. Now fix $\alpha = \zeta_j$ and consider

$$W(\zeta)W(\zeta_j) - W(\zeta_j)W(\zeta)$$

which is a holomorphic function of ζ in Z vanishing for $\zeta = \zeta_k$, $k = 0, 1, 2, \cdots$. By Theorem 3.10.4 the difference then has to vanish identically in Z. Next we fix ζ in Z and consider the difference

$$W(\zeta)W(\alpha) - W(\alpha)W(\zeta)$$

which is a holomorphic function of α in Z vanishing for $\alpha = \zeta_k$, $k = 0, 1, 2, \cdots$. Consequently

$$W(\zeta)W(\alpha) \equiv W(\alpha)W(\zeta), \qquad a, \zeta \in Z. \tag{13.2.3}$$

The next step of the proof is the establishing of the formula

$$W(\zeta) = W(\zeta - \alpha)W(\alpha) \tag{13.2.4}$$

for all ζ and α such that ζ, α, and $\zeta - \alpha$ are in Z. The right-hand side is well defined for $\alpha \in Z$, $\zeta - \alpha \in Z$ and is holomorphic in each variable separately. We can find three associated circles in Z, say $\gamma_1 : |\zeta - \zeta_0| = \rho_1$, $\gamma_2 : |\zeta - \zeta_0| = \rho_2$, and $\gamma_3 : |\zeta - 2\zeta_0| = \rho_3$ with $\rho_2 = \rho_1 + \rho_3$ such that if α is in γ_1 and ζ is in γ_3 then $\zeta - \alpha$ is in γ_2. Give α the value ζ_j where $j > j_0$ is so large that ζ_j is in γ_1. Then $W(\zeta) - W(\zeta - \zeta_j)W(\zeta_j)$ is holomorphic for ζ in γ_3 and vanishes for all points $\zeta = \zeta_j + \zeta_k$, $k > k_0$. Theorem 3.10.4 applies and shows that (13.2.4) is true for $\alpha = \zeta_j$ and ζ in the convex domain $Z \cap Z + \zeta_j$. Here $Z + \zeta_j$ is the image of Z under the translation $\zeta' = \zeta + \zeta_j$. We now give ζ a fixed value ζ' in γ_3 and consider $W(\zeta' - \alpha)W(\alpha)$ which is a holomorphic function of α in $Z \cap \zeta' - Z$. This domain contains the interior of γ_1. For $\alpha = \zeta_j$, $j > j_0$, the function has the constant value $W(\zeta')$; by Theorem 3.10.4 it is identically equal to $W(\zeta')$ for $\alpha \in Z \cap \zeta' - Z$. Fixing α in γ_1, we see that (13.2.4) holds for all ζ in γ_3 and hence for all ζ in $Z \cap Z + \alpha$. By analytic continuation with respect to α, we see that the relation holds for all ζ and α such that ζ, α, $\zeta - \alpha$ are in Z. It follows that (13.2.2) holds for ζ', ζ'', $\zeta' + \zeta''$ in Z.

The final step in the proof is the extension of $W(\zeta)$ from Z to $(Z)_a$. We start by defining $W(\zeta)$ in the convex domain Z_2 by

$$W(\zeta) = [W(\zeta/2)]^2. \tag{13.2.5}$$

Since $\zeta/2 \in Z$, the definition makes sense and the function $W(\zeta)$ is holomorphic in Z_2. A simple calculation shows that (13.2.2) holds for ζ', ζ'', $\zeta' + \zeta''$ in Z_2. The convex domains Z and Z_2 intersect in a convex domain containing the point $2\zeta_0$. For ζ in $Z \cap Z_2$ we have $[W(\zeta/2)]^2 = W(\zeta)$ by (13.2.4) as well as by (13.2.5) so the two definitions agree. Thus $W(\zeta)$ is now defined as a holomorphic function of ζ in $Z \cup Z_2$ and satisfies (13.2.2) for ζ', ζ'', $\zeta' + \zeta''$ in this domain.

We then define $W(\zeta)$ step by step in $Z_3, Z_4, \cdots$ by

$$W(\zeta) = [W(\zeta/n)]^n, \qquad \zeta \in Z_n. \tag{13.2.6}$$